The structure of Zeckendorf representations and base φ expansions

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One World Seminar on Combinatorics on Words

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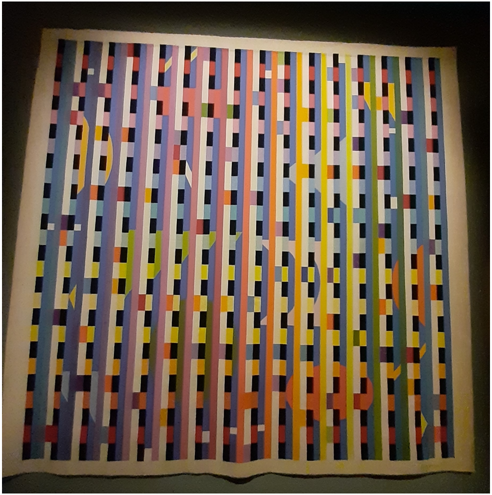
Short Abstract: In the Zeckendorf numeration system natural numbers are represented as sums of Fibonacci numbers. In base φ natural numbers are represented as sums of powers of the golden mean φ. Both representations have digits 0 and 1, where the word 11 is not allowed. I will *try to* answer the following questions: what are the words that can occur in the Zeckendorf representations, and what are those that occur in base φ expansions? In which representations, c.q. expansions, of which natural numbers do they occur?

At the museum



Yaakov Agam: “The image needs to evolve, not exist”

Zeckendorf representations

Let $F_0 = 0,\ F_1 = 1,\ F_2 = 1, \ldots$ be the Fibonacci numbers.

Ignoring leading zeros, any natural number N can be written uniquely as

$$N = \sum_{i=0}^{\infty} d_i F_{i+2},$$

with digits $d_i = 0$ or 1, and where $d_i d_{i+1} = 11$ is not allowed.

We write $Z(N) = d_L \ldots d_2 d_1 d_0$.

Example $Z(6) = 1001$, since $F_5 = 5, F_2 = 1$.

Base phi expansions

Base phi expansions are also known as beta-expansions, with $\beta = (1+\sqrt{5})/2 =: \varphi$, the golden mean.

A natural number N is written in base phi if N has the form

$$N = \sum_{i=-\infty}^{\infty} d_i \varphi^i,$$

with digits $d_i = 0$ or 1, and where $d_i d_{i+1} = 11$ is not allowed. Similarly to base 10 numbers, we write

$$\beta(N) = d_L d_{L-1} \dots d_1 d_0 \cdot d_{-1} d_{-2} \dots d_{R+1} d_R.$$

Example $\beta(5) = 1000 \cdot 1001$, since $\varphi^3 + \varphi^{-1} + \varphi^{-4} = 5$.

Zeckendorf and base phi

N	$Z(N)$	$\beta(N)$
1	1	$1\cdot$
2	10	$10 \cdot 01$
3	100	$100 \cdot 01$
4	101	$101 \cdot 01$
5	1000	$1000 \cdot 1001$
6	1001	$1010 \cdot 0001$
7	1010	$10000 \cdot 0001$
8	10000	$10001 \cdot 0001$
9	10001	$10010 \cdot 0101$
10	10010	$10100 \cdot 0101$
11	10100	$10101 \cdot 0101$
12	10101	$100000 \cdot 101001$
13	100000	$100010 \cdot 001001$
14	100001	$100100 \cdot 001001$
15	100010	$100101 \cdot 001001$

Main differences:

a) Shift invariance for $Z(\cdot)$

b) real numbers for $\beta(\cdot)$

Zeckendorf and base phi, part 2

There is a paper which describes a two-tape automaton with
input: the Zeckendorf representation
output: the base phi expansion.

C. Frougny and J. Sakarovitch, Automatic conversion from Fibonacci representation to representation in base φ and a generalization. Int. J. Algebra Comput. 9 (1999)

A sea of words

[001000001010·00001010010]

[001000010000·00001010010]

[001000010001·00001010010]

[001000010010·01001010010]

[001000010100·01001010010]

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[001001000010·01000010010]

[001001000100·01000010010]

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Sum of digits for Zeckendorf

For $Z(N) = d_L \dots d_1 d_0$, let $s_Z(N) := d_L + \dots + d_1 + d_0 \mod 2$.

$$s_Z = 0, 1, 1, 1, 0, 1, 0, 0, 1, 0, 0, 0, 1, 1, 0, 0, 0, 1, 0, 1, 1, 1, 0, \dots$$

Then s_Z is a morphic sequence.

$$\theta_Z := \quad 1 \to 12,\ 2 \to 4,\ 3 \to 1,\ 4 \to 43,$$

$$\lambda := \quad 1 \to 0,\ 2 \to 1,\ 3 \to 0,\ 4 \to 1.$$

$x = 1244343\dots$ with $\theta_Z(x) = x$, then $\lambda(x) = s_Z$.

J.-P. Allouche and J. Shallit, Automatic Sequences (2003), Examples 7.8.2 and 7.8.4. On 6 letters.

E. Ferrand, An analogue of the Thue-Morse sequence, The Electronic Journal of Combinatorics (2007)

Complexity of the Zeckendorf fixed point

The Zeckendorf fixed point is the fixed point
$x_Z = 12443431431\ldots$ of the morphism
$\theta_Z := \quad 1 \to 12,\ 2 \to 4,\ 3 \to 1,\ 4 \to 43.$

Let $p = (p(n))$ be the subword complexity function of x_Z.

We have $p(1) = 4, p(2) = 10, p(3) = 16, p(4) = 22, p(5) = 28.$

Let ff be the infinite word on the alphabet $\{6, 8\}$ given by

$$ff = 686688666888\cdots = 6^{F_2}8^{F_2}6^{F_3}8^{F_3}6^{F_4}8^{F_4}\ldots.$$

Conjecture 1 $p(n+5) - p(n+4) = ff(n)$ for $n = 1, 2\ldots.$

S. Brlek, Enumeration of factors in the Thue–Morse word, Discrete Appl. Math.(1989)

Complexity of Zeckendorf sum of digits mod 2

Let $p = (p(n))$ be the subword complexity function of s_Z.
We have $p(1) = 2, p(2) = 4, p(3) = 8, p(4) = 14, p(5) = 24$.

Let xf be the infinite word on the alphabet $\{6, 8\}$ given by

$$xf = 6686886666888 \cdots = 6^{X_2} 8^{F_2} 6^{X_3} 8^{F_3} 6^{X_4} 8^{F_4} \ldots.$$

Here $X_2 = 2, X_3 = 1, X_4 = 4, X_5 = 4, \cdots$:
X_n is the absolute value of the Euler characteristic of the Boolean complex of the Coxeter group A_n $\boxed{\text{:-)}}$

$$X_n := F_n + (-1)^n$$

Conjecture 2 $p(n + 5) - p(n + 4) = xf(n)$ for $n = 1, 2 \ldots.$

Sum of digits for base phi

For $\beta(N) = d_L d_{L-1} \dots d_0 \cdot d_{-1} \dots d_{R+1} d_R$, let

$$s_\beta(N) := d_L + \cdots + d_0 + d_{-1} + \cdots + d_R \mod 2.$$

$$s_\beta = 0, 1, 0, 0, 1, 1, 1, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1, \dots$$

Then s_β is a morphic sequence.

$$\begin{array}{llll} \tau(1) = 12, & \tau(2) = 312, & \tau(3) = 47, & \tau(4) = 8312, \\ \tau(5) = 56, & \tau(6) = 756, & \tau(7) = 83, & \tau(8) = 4756. \end{array}$$

$$\lambda(1) = \lambda(3) = \lambda(6) = \lambda(8) = 0, \; \lambda(2) = \lambda(4) = \lambda(5) = \lambda(7) = 1.$$

$t = 1231247123\dots$ with $\tau(t) = t$, then $s_\beta = \lambda(t)$.

M.D., The sum of digits function of the base phi expansion of the natural numbers, INTEGERS (2020)

Pseudo randomness

Let s_2 be the Thue Morse sequence.

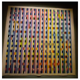
Michael Drmota, Christian Mauduit and Joël Rivat:

Theorem The sequence $(s_2(n^2))$ is a normal sequence.

Conjecture 3 The sequence $(s_Z(n^2))$ is a normal sequence.

Conjecture 4 The sequence $(s_\beta(n^2))$ is a normal sequence.

M. Drmota, C. Mauduit, and J. Rivat, Normality along squares, J. Eur. Math. Soc. (2019)



Beatty sequences

Beatty sequence: $A(N) = \lfloor N\alpha \rfloor$ for $N \geq 1$, where α is a positive real number.

Beatty observed: if $B(N) := \lfloor N\beta \rfloor$, with

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1,$$

then $(A(N))$ and $(B(N))$ are *complementary* sequences.

The golden mean case: Wythoff sequences

Lower Wythoff sequence:

$$(A(N)) = (\lfloor N\varphi \rfloor) = (1, 3, 4, 6, 8, 9, 11, \dots),$$

Upper Wythoff sequence:

$$(B(N)) = (\lfloor N\varphi^2 \rfloor) = (2, 5, 7, 10, 13, 15, \dots),$$

$$\frac{1}{\varphi} + \frac{1}{\varphi^2} = 1.$$

Compound Wythoff sequences

An important role is played by compositions of the two sequences A and B , also known as *compound Wythoff sequences*.

As usual, we write these compositions as words over the monoid generated by A, B. For example, the compound sequence AB is given by

$$AB(N) = A(B(N)) \quad N = 1, 2 \ldots.$$

Generalized Beatty sequences

Let α be an irrational number larger than 1.

Generalized Beatty sequence V:

$$V(N) = p\lfloor N\alpha \rfloor + qN + r,\ \ N \geq 1.$$

p, q and r integers, the *parameters* of V.

J.-P. Allouche and F.M. Dekking, Generalized Beatty sequences and complementary triples, Moscow J. Comb. Number Th. (2019)

Generalized Beatty sequences, Part 2

Lemma Let V be a generalized Beatty sequence with parameters (p, q, r), and $\alpha = \varphi$. Then VA and VB are generalized Beatty sequences with parameters

$(p_{VA}, q_{VA}, r_{VA}) = (p + q, p, r - p)$,

$(p_{VB}, q_{VB}, r_{VB}) = (2p + q, p + q, r)$.

Example The Wythoff sequence

$$(A(N)) = (\lfloor N\varphi \rfloor) = (1, 3, 4, 6, 8, 9, 11, 12, 14, \dots),$$

is a GBS with parameters $(1, 0, 0)$.
The iterated Wythoff sequence $AA = (1, 4, 6, 9, 12, 14, 17, \dots)$ is a GBS with parameters $(1, 1, -1)$.

J.-P. Allouche and F.M. Dekking, Generalized Beatty sequences and complementary triples, Moscow J. Comb. Number Th. (2019)

Zeckendorf: technical detail

N in $\{0, \ldots, F_n - 1\}$: supplement with 0's $Z(N) \Rrightarrow Z^*(N)$.

For example, for $n = 6$, we have

N	$Z(N)$	$Z^*(N)$
1	1	00001
2	10	00010
3	100	00100
4	101	00101
5	1000	01000
6	1001	01001
7	1010	01010
8	10000	10000

In the following, occurrences of a word w have to be interpreted in the Z^*-sense.

Zeckendorf structure

For any natural number m fix a word $w = w_{m-1} \dots w_0$ of 0's and 1's.

We are interested in the numbers N with $Z(N) = d_L \dots d_2 d_1 d_0(N)$ such that

$$d_{m-1} \dots d_0(N) = w_{m-1} \dots w_0.$$

We write R_w for the sequence of occurrences of those N.

For example, $R_{010} = (2, 7, 10, 15, 20, \dots)$.

It turns out that the sequences R_w are always generalized Beatty sequences, and almost always compound Wythoff sequences, which we denote by C_w.

Some results from the literature

In a pioneering paper by Carlitz, Scoville and Hoggatt, we find that for $m \geq 0$

$$\begin{aligned} C_{10^{2m+1}} &= B^{m+1}A, & C_{10^{2m}} &= AB^mA, \\ C_{0010^{2m+1}} &= B^{m+1}AA, & C_{010^{2m}} &= AB^mAA, \\ C_{1010^{2m+1}} &= B^{m+1}AB, & C_{1010^{2m}} &= AB^mAB. \end{aligned}$$

These are given in their Theorems 7 and 8.

L. Carlitz, R. Scoville, V. E. Hoggatt, Jr., Fibonacci representations, Fibonacci Quart. (1972).

Key lemma

Lemma For any natural number $m > 1$ fix a word $w = w_{m-1} \dots w_0$ of 0's and 1's, with $w_{m-1} = 0$.

Let C_w be the Wythoff-coding of the sequence of occurrences of the numbers N whose Z^*-expansion ends with w. Then

$$C_{0w} = C_w A, \quad C_{1w} = C_w B.$$

This would have been very useful to L. Carlitz, R. Scoville, V. E. Hoggatt, Jr.,....

Zeckendorf: main result

Theorem For any natural number m fix a word $w = w_{m-1} \dots w_0$ of 0's and 1's, containing no 11. Then—except if $w = 1$, or $w = 0^m$—the sequence R_w of occurrences of numbers N such that the m lowest digits of the Zeckendorf expansion of N are equal to w, i.e., $d_{m-1} \dots d_0 = w$, is a compound Wythoff sequence C_w.

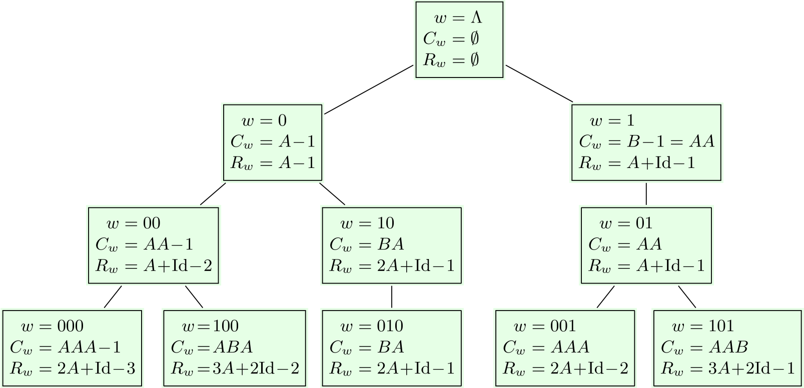
For *all* w:

$$R_w = F_m A + F_{m-1}\mathrm{Id} + \gamma_w \quad \text{if } w_{m-1} = 0$$
$$R_w = F_{m+1} A + F_m\mathrm{Id} + \gamma_w \quad \text{if } w_{m-1} = 1$$

for some negative integer γ_w.

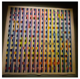
Exceptional cases: $R_1 = B - 1$; $R_{0^m} = A^m - 1$.

Zeckendorf blocks on the Fibonacci tree



$(p_{VA}, q_{VA}, r_{VA}) = (p+q, p, r-p)$,

$(p_{VB}, q_{VB}, r_{VB}) = (2p+q, p+q, r)$.



What about base phi expansions?

A natural number N is written in base phi if N has the form

$$N = \sum_{i=-\infty}^{\infty} d_i \varphi^i,$$

with digits $d_i = 0$ or 1, and where $d_i d_{i+1} = 11$ is not allowed.

$$\beta(N) = d_L d_{L-1} \dots d_1 d_0 \cdot d_{-1} d_{-2} \dots d_{R+1} d_R.$$

$$\beta(N) = \beta^+(N) \cdot \beta^-(N).$$

Treat $\beta^+(N)$ and $\beta^-(N)$ separately.

Base phi

N	$\beta(N)$	$T(N)$
1	1	C
2	$10 \cdot 01$	A
3	$100 \cdot 01$	B
4	$101 \cdot 01$	C
5	$1000 \cdot 1001$	D
6	$1010 \cdot 0001$	A
7	$10000 \cdot 0001$	B
8	$10001 \cdot 0001$	C
9	$10010 \cdot 0101$	A
10	$10100 \cdot 0101$	B
11	$10101 \cdot 0101$	C
12	$100000 \cdot 101001$	D
13	$100010 \cdot 001001$	A
14	$100100 \cdot 001001$	B
15	$100101 \cdot 001001$	C
16	$101000 \cdot 100001$	D
17	$101010 \cdot 000001$	A
18	$1000000 \cdot 000001$	B
19	$1000001 \cdot 000001$	C
20	$1000010 \cdot 010001$	A

Coding:

$T(N) = \mathrm{A}$ iff $d_1 d_0 \cdot d_{-1}(N) = 100$,

$T(N) = \mathrm{B}$ iff $d_1 d_0 \cdot d_{-1}(N) = 000$,

$T(N) = \mathrm{C}$ iff $d_1 d_0 \cdot d_{-1}(N) = 010$,

$T(N) = \mathrm{D}$ iff $d_1 d_0 \cdot d_{-1}(N) = 001$.

Lemma:

$d_1 d_0 \cdot d_{-1}(N) = 101$ never occurs.

A,B,C,D,...

Let γ on the alphabet $\{\mathrm{A},\mathrm{B},\mathrm{C},\mathrm{D}\}$ be defined by:

$$\gamma(\mathrm{A}) = \mathrm{AB}, \quad \gamma(\mathrm{B}) = \mathrm{C}, \quad \gamma(\mathrm{C}) = \mathrm{D}, \quad \gamma(\mathrm{D}) = \mathrm{ABC}.$$

Theorem The sequence $(T(N))_{N\geq 2}$ is the unique fixed point of the morphism γ.

Observe: $\gamma(\mathrm{ABC}) = \mathrm{ABC\,D}$, $\gamma(\mathrm{D}) = \mathrm{ABC}$.

M.D., Base phi representations and golden mean beta-expansions,
Fib. Quarterly 58 (2020)

Base phi and Lucas numbers

The Lucas numbers
$(L_n) = (2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, \dots)$:

$$L_0 = 2, \quad L_1 = 1, \quad L_n = L_{n-1} + L_{n-2} \quad \text{for } n \geq 2.$$

From $L_{2n} = \varphi^{2n} + \varphi^{-2n}$, and $L_{2n+1} = L_{2n} + L_{2n-1}$:

$$\beta(L_{2n}) = 10^{2n} \cdot 0^{2n-1}1, \quad \beta(L_{2n+1}) = 1(01)^n \cdot (01)^n.$$

Partition the natural numbers into Lucas intervals:

$$\Lambda_{2n} := [L_{2n},\, L_{2n+1}] \quad \text{and} \quad \Lambda_{2n+1} := [L_{2n+1} + 1,\, L_{2n+2} - 1].$$

Divide the interval $\Lambda_{2n+1} = [L_{2n+1} + 1, L_{2n+2} - 1]$ into three parts:

$$I_n := [L_{2n+1} + 1,\, L_{2n+1} + L_{2n-2} - 1],$$
$$J_n := [L_{2n+1} + L_{2n-2},\, L_{2n+1} + L_{2n-1}],$$
$$K_n := [L_{2n+1} + L_{2n-1} + 1,\, L_{2n+2} - 1].$$

Recursive Structure Theorem

Theorem

I For all $n \geq 1$ and $k = 1, \ldots, L_{2n-1}$ one has
$\beta(L_{2n} + k) = \beta(L_{2n}) + \beta(k) = 10\ldots0\,\beta(k)\,0\ldots01.$

II For all $n \geq 2$ and $k = 1, \ldots, L_{2n-2} - 1$ one has

$$I_n: \quad \beta(L_{2n+1} + k) = 1000(10)^{-1}\beta(L_{2n-1} + k)(01)^{-1}1001,$$

$$K_n: \quad \beta(L_{2n+1} + L_{2n-1} + k) = 1010(10)^{-1}\beta(L_{2n-1} + k)(01)^{-1}0001.$$

Moreover, for all $n \geq 2$ and $k = 0, \ldots, L_{2n-3}$

$$J_n: \quad \beta(L_{2n+1} + L_{2n-2} + k) = 10010(10)^{-1}\beta(L_{2n-2} + k)(01)^{-1}001001.$$

History of the Recursive Structure Theorem

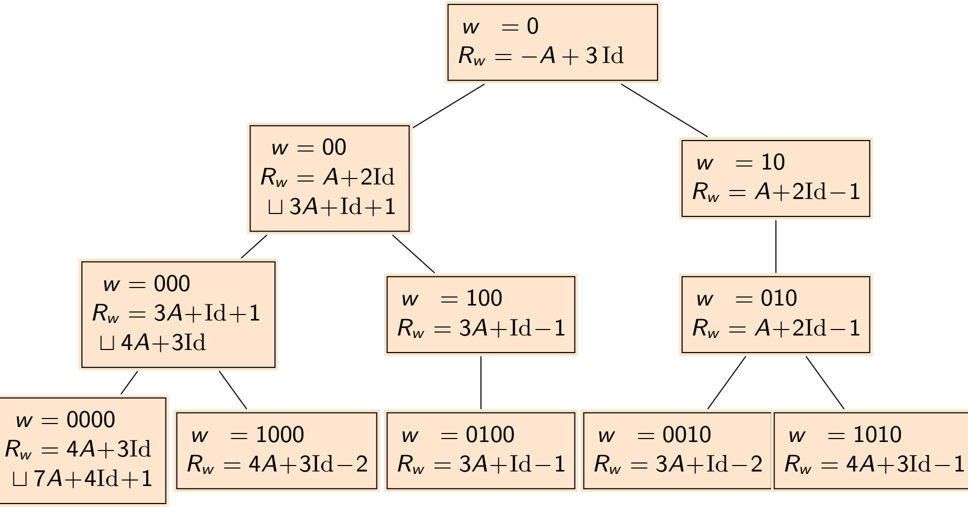
E. Hart, On Using Patterns in the Beta-Expansions To Study Fibonacci-Lucas Products, Fibonacci Quart. 36 (1998), 396–406.

E. Hart and L. Sanchis, On the occurrence of F_n in the Zeckendorf decomposition of nF_n, Fibonacci Quart. 37 (1999), 21–33.

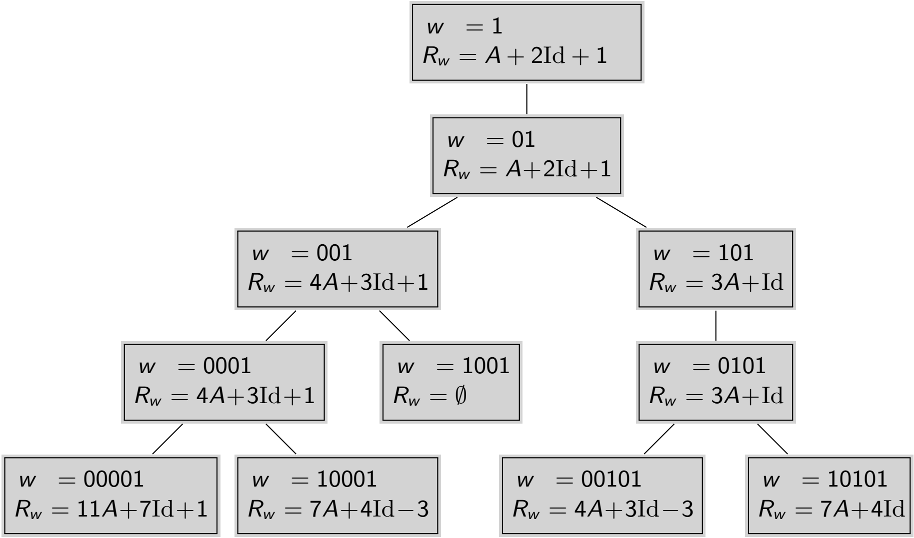
G.R. Sanchis and L.A. Sanchis, On the frequency of occurrence of α^i in the α-expansions of the positive integers, Fibonacci Quart. 39 (2001), 123–137.

M.D. How to add two natural numbers in base phi. To appear in Fib. Quarterly (2020).

Digit blocks $w = d_m d_{m-1} \dots d_1 0$



Digit blocks $w = d_m d_{m-1} \dots d_1 1$



The missing blocks

Theorem For any natural number m fix a word w of 0's and 1's, containing no 11. Let $w_0 = 1$. Then the sequence of occurrences of numbers N such that the digits $d_{m-1} \dots d_0$ of the base phi expansion of N are equal to w, i.e.,

$$d_{m-1} \dots d_0(N) = w,$$

is a generalized Beatty sequence, *with exception* of the words w with suffix $10^{2m}1$, for $m = 2, 3, \dots,$ which do not occur at all.

M.D., Work in progress (2020)

The end