AUTOMATIC GROUPS AND AMALGAMS

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Part 1: Introduction

1. Basic objectives and explanatory remarks.

Automatic and aysnchronously automatic groups were invented a few years ago by J.W. Cannon, D.B.A. Epstein, D.F. Holt, M.S. Paterson and W.P. Thurston in [**CEHPT**].

The primary objective of this paper is to report on a number of new results about the construction of new automatic and asynchronously automatic groups from old by means of amalgamated products. A survey of these results will appear in [BGSS2].

In order to keep this paper as self-contained as possible, we have provided here a leisurely and mainly self-contained exposition of a little of the theory of automatic groups. This exposition is based in large measure on the pioneering work of Cannon, Epstein, Holt, Paterson and Thurston, which as of now exists only in preprint form [**CEHPT**].

Consequently this paper is divided up into three parts. The first one is an introduction to our new work, the second is the exposition alluded to above of part of the theory of automatic groups and the third part is devoted to our work on amalgamated products of automatic groups.

Much of this work was done while the four authors were participants in the year long conference on geometric methods in combinatorial group thoery at MSRI during the fall of 1988. Subsequently Gersten and Short (jointly) and Shapiro (in some independent work) have developed some new ideas and perspectives for studying automatic groups. In particular Gersten and Short have utilised and extended some work of Gilman [**Gi**] in [**GS3**]. Shapiro [**S**] has studied fundamental groups of finite graphs of automatic groups and has obtained a version of our Theorem A for finite graph products. Many of their ideas have been incorporated into the present version of our earlier work which has appeared in preliminary preprint form as [**BGSS1**].

We would like to thank John Stallings for a number of insightful comments and also Frank Rimlinger for carefully reading and commenting on [**BGSS1**].

2. Automatic groups.

Let $\mathcal{A}$ be the finite set $\{a_1, \ldots, a_q\}$ and let $\mathcal{A}^\star$ be the set of all *strings* or *words*

$$w = b_1 \ldots b_n$$

made up from *letters* b_i in $\mathcal{A}$. $\mathcal{A}^\star$ includes the empty word which we denote by e. We term n *the length of* w, which we denote by $\ell(w)$. As usual $\mathcal{A}^\star$ can be turned into a monoid by equipping it with the binary operation concatenation (i.e. juxtaposition). The subsets of $\mathcal{A}^\star$ are often referred to as *languages over* $\mathcal{A}$. We shall be concerned with special languages over $\mathcal{A}$ termed *regular languages over* $\mathcal{A}$ or sometimes *regular sets (over* $\mathcal{A}$). We take for granted here the definition of such a regular set (see II.A.1 for the definition). We will also need to consider regular sets over the alphabet

$$\mathcal{A}(2,\$) = (\mathcal{A} \cup \{\$\}) \times (\mathcal{A} \cup \{\$\}) - \{(\$,\$)\}.$$

Here \$ is a so-called *padding symbol* which we assume not to lie in the set $\mathcal{A}$. $\mathcal{A}(2,\$)$ arises from a consideration of pairs (u, v) of words $u, v \in \mathcal{A}^\star$. The point here is that although $\mathcal{A}^\star \times \mathcal{A}^\star$ can be turned into a monoid in the natural way via coordinatewise multiplication it is not generated by $\mathcal{A} \times \mathcal{A}$. This means that there is no obvious way of expressing (u, v) as a product of pairs (a, b) of letters $a, b \in \mathcal{A}$ when u and v have

different lengths. The introduction of the \$ symbol is a technical device designed to get around this difficulty. In fact we invoke here a mapping ν from $\mathcal{A}^\star \times \mathcal{A}^\star$ into $\mathcal{A}(2, \$)^\star$ which is defined as follows. If $u = a_1 \dots a_n, v = b_1 \dots b_m$, then

$$\nu : (u, v) \mapsto (a_1, b_1) \dots (a_m, b_m)(a_{m+1}, \$) \dots (a_n, \$) \ if \ m < n,$$

$$\nu : (u, v) \mapsto (a_1, b_1) \dots (a_n, b_n) \ if \ m = n,$$

$$\nu : (u, v) \mapsto (a_1, b_1) \dots (a_n, b_n)(\$, b_{n+1}) \dots (\$, b_m) \ if \ m > n$$

and

$$\nu : (e, e) \mapsto e.$$

Now let G be a group. Then we term $\mathcal{A}$ *a set of monoid generators of* G if it comes equipped with a map, termed *a monoid generating map*,

$$a \mapsto \overline{a} \ (a \in \mathcal{A}, \ \overline{a} \in G)$$

of $\mathcal{A}$ into G whose extension μ to a monoid homomorphism of $\mathcal{A}^\star$ into G is a surjection. This means that every element of G can be expressed as a product of the elements in $\{\overline{a_1}, \dots, \overline{a_q}\}$. The map $a \mapsto \overline{a}$ need not be, but often is, monic, indeed even an inclusion. However it is important to note, in order to avoid confusion, that in many cases $\mathcal{A}$ is not a subset of G. The image of $w \in \mathcal{A}^\star$ under μ is usually denoted here by $\overline{w}$.

Suppose again that G is a group and that $\mathcal{A}$ is a finite set of monoid generators of G. A regular subset L of $\mathcal{A}^\star$ which maps surjectively to G via the map μ is termed a *regular language (over* $\mathcal{A}$*) for* G and the pair $(\mathcal{A}, L)$ *a rational structure for* G. The word $w \in L$ is referred to as a *representative of* $g \in G$ if $\overline{w} = g$. If every element of G has exactly one representative then we will refer to $(\mathcal{A}, L)$ as a *rational structure with uniqueness for* G (Gilman [Gi] calls this a rational cross section for G).

Suppose that $(\mathcal{A}, L)$ is a rational structure for the group G. We need to consider the following subsets of $\nu(\mathcal{A}^\star \times \mathcal{A}^\star)$:

$$L_= = \{\nu(w_1, w_2) \mid w_1, w_2 \in L, \overline{w_1} = \overline{w_2}\},$$

and, for each $w \in \mathcal{A}^\star$ the set

$$L_w = \{\nu(w_1, w_2) \mid w_1, w_2 \in L, \overline{w_1} = \overline{w_2 w}\}.$$

In the event that $w = a_i \in \mathcal{A}$ we sometimes denote L_w by L_i.

The group G is termed an *automatic group* if there exists a rational structure $(\mathcal{A}, L)$ for G with the following properties:

(1) $L_=$ is regular;
(2) L_{a_i} is regular for each $i = 1, 2, \dots, q$.

We term such a rational structure an *automatic structure* or *an automatic structure over* $\mathcal{A}$. Thus a group G is automatic if it has an automatic structure.

3. Some properties of automatic groups.

As we have already noted, automatic groups were invented by J.W. Cannon, D.B.A. Epstein, D.F. Holt, M.S. Paterson and W.P. Thurston in [**CEHPT**], although the notion was hinted at in earlier work of J.W. Cannon [**C**]. They have proved that if G is an automatic group and if $\mathcal{B}$ is *any* finite set of monoid generators for G, then there is an automatic structure for G over $\mathcal{B}$ ([CEHPT]). Thus *the property of being automatic is independent of the choice of the finite set of monoid generators.*

The very definition of an automatic group suggests that its structure is reasonably uncomplicated. This is manifested by the fact that *automatic groups are finitely presented* and also that they *have a solvable word problem.* These results are due to Cannon et al. [**CEHPT**]. It is worth noting, however, that the class of automatic groups is quite extensive. For instance it includes the so-called *hyperbolic* groups of M. Gromov [**Gr**], which, following a suggestion of Cannon, we shall term here *negatively curved* groups and which will be defined below. In particular *the fundamental groups of closed hyperbolic manifolds are negatively curved* (Gromov [**Gr**]) and *hence are automatic.* Moreover Cannon et al. [**CEHPT**] have proved that many common finitely presented groups are automatic; for example all *finite groups*, all *finitely generated free groups* and all *finitely generated abelian groups* are automatic. More generally they have shown that *the class of automatic groups is closed under finite free products, finite direct products and finite extensions* but not under finitely generated subgroups. Somewhat surprisingly *a finitely generated nilpotent group is automatic if and only if it contains a subgroup of finite index which is abelian* [**CEHPT**]. Recently Gersten and Short [GS1], [GS2] have shown that *all of the non-metric small cancellation groups of Lyndon* [LS] *are also automatic.*

It is clear then that automatic groups constitute a very interesting class of groups.

As we have already seen, an automatic group is described in terms of a regular language over some alphabet. The very definition of a regular set implies that there is a finite state automaton which recognizes this regular language (see II.A.1). Epstein and Rees [ER] have written computer programs to compute such a finite state automaton. Thus it is in some sense practical to use computers to compute "products" in automatic groups. This means that Epstein and Rees have built a bridge between combinatorial group theory and computer science, which may well have very interesting consequences.

4. Asynchronously automatic groups.

Let $\mathcal{A}$ be as usual a finite alphabet. Rabin and Scott [RS] have introduced the notion of a *two-tape* or *asynchronous automaton*, which makes it possible to deal directly with subsets of $\mathcal{A}^\star \times \mathcal{A}^\star$, without introducing a padding symbol. This leads to the definition of an *asynchronously regular* subset of $\mathcal{A}^\star \times \mathcal{A}^\star$, which we will discuss in detail in II.A.6 together with the notion of a two-tape automaton. It allows one to define, following Epstein, Cannon, Holt, Paterson and Thurston [CEHPT], an *asynchronously automatic group.* The definition is analogous to that of an automatic group. More precisely the group G is termed *asynchronously automatic* if it has a rational structure $(\mathcal{A}, L)$ such that the sets

$$L_{(=)} = \{(w_1, w_2) \mid w_1, w_2 \in L, \overline{w_1} = \overline{w_2}\},$$

and

$$L_{(i)} = \{(w_1, w_2)) \mid w_1, w_2 \in L, \overline{w_1} = \overline{w_2 a_i}\} \ (a_i \in \mathcal{A})$$

are all asynchronously regular. We term such a rational structure *an asynchronously automatic structure for G.*

Now if K is a subset of $\mathcal{A}^\star \times \mathcal{A}^\star$ and $\nu(K)$ is regular, then K is asynchronously regular. Consequently the *class of automatic groups is contained in the class of asynchronously automatic groups.* It also turns out, as in the case of automatic groups, that *asynchronously automatic groups are finitely presented* and *have solvable word problem* ([**CEHPT**]. They play a very important role in our work.

5. Cayley graphs and fellow travellers.

Let G be a group and let $\mathcal{X}$ be a set. Then we term $\mathcal{X}$ *a set of group generators of G* if it comes equipped with a map

$$x \mapsto \overline{x}$$

of $\mathcal{X}$ into G, termed a *group generation map*, such that its extension μ to a homomorphism of the free group F on $\mathcal{X}$ into G is a surjection. We now adjoin to $\mathcal{X}$ a set $\mathcal{X}^{-1} = \{x^{-1} \mid x \in \mathcal{X}\}$ in one-to-one correspondence with $\mathcal{X}$ and put $\mathcal{A} = \mathcal{X} \cup \mathcal{X}^{-1}$. Let us now extend the group generation map $x \mapsto \overline{x}$ $(x \in \mathcal{X})$ to a monoid homomorphism again denoted μ, of $\mathcal{A}^\star$ into G by sending x^{-1} to $\overline{x}^{-1}$. We denote the image of $w \in \mathcal{A}^\star$ under μ by $\overline{w}$ and we term w *reduced* if it does not contain a subword of the form xx^{-1} or $x^{-1}x$. We will call a set $\mathcal{X}$ of group generators *closed under inverses* if it is equipped with an
involution $\sigma : \mathcal{X} \to \mathcal{X}$ such that for each $x \in \mathcal{X}$, $\overline{\sigma(x)} = \overline{x}^{-1}$. Clearly, $\mathcal{A}$ as defined above is closed under inverses.

We now define the Cayley graph $\Gamma = \Gamma(G) = \Gamma_{\mathcal{X}}(G)$ of G relative to the set $\mathcal{X}$ of group generators of G. Γ is a directed graph with vertex set G and edges all triples $(g, a, g\overline{a})$, where $g \in G, a \in \mathcal{A}$. We respectively term g the *origin*, h the *terminus* and a the *label* of the edge (g, a, h). We also sometimes refer to the edges $(g, a, g\overline{a})$ and $(g\overline{a}, a^{-1}, g)$ as *inverses.* The origin and terminus of an edge are referred to as its *extremities.* We term a sequence γ of (not necessarily distinct) vertices $g_0, \ldots, g_n$ of Γ *a path of length* n if either $n = 0$ or in the case where $n > 0$, if for each $i = 0, \ldots, n-1$ either $g_i = g_{i+1}$ or there exists an edge whose origin is g_i and whose terminus is g_{i+1}. We term g_0 the *origin* and g_n the *terminus* of γ and refer to them as the *extremities* of γ and we say that γ *goes* from g_0 to g_n. We will also use the term *path* to refer to an infinite sequence, $g_0, \ldots$ of vertices such that for each $i \geq 0$, either $g_i = g_{i+1}$ or there is a directed edge whose origin is g_i and whose terminus is g_{i+1}. As above, g_0 is the *origin* of such a path.

The Cayley graph is always path connected.

If P and Q are vertices in a Cayley graph $\Gamma_{\mathcal{X}}$, then the distance $d(P, Q)$ between them is defined to be the minimum length of a path from P to Q. Since $\mathcal{A}$ is closed under inverses, for each path that goes from P to Q, there is a corresponding path that goes from Q to P. This turns Γ into a metric space. The restriction of this metric to vertices gives an integer valued metric. We shall sometimes refer to the vertices of $\Gamma_{\mathcal{X}}(G)$, i.e. to the elements of G as *points.* A shortest path from the group element g to the group element h is termed a *geodesic* and a "triangle" in $\Gamma_{\mathcal{X}}(G)$ is termed *a geodesic triangle* if its sides are geodesics. Gromov [Gr] has termed a geodesic triangle $\delta - thin$ if every point on one side of the triangle is no further than δ from at least one point on one of the other two sides, i.e. each side of the triangle is contained in a δ-neighbourhood of the union of the other two sides. The group G is then termed *negatively curved* if there exists a δ such that every geodesic triangle in $\Gamma_{\mathcal{A}}(G)$ is δ-thin.

As already noted above, these negatively curved groups were introduced by Gromov [Gr]. They can be described rather differently by means of so-called *isoperimetric inequalities.* To this end suppose that $\langle X; R\rangle$ is a presentation of the group G. Thus X comes equipped with a generation map whose extension to the free group F on $\mathcal{X}$ is a homomorphism μ of F onto G with kernel K the normal closure in F of R. We say that $f : \mathbb{N} \to \mathbb{R}$ is a *Dehn function* for this presentation of G if it satisfies the following condition:
for any freely reduced word w, viewed as an element of F, such that $\overline{w} = 1$ *there are words $r_i \in R$, $p_i \in F$ and $\epsilon_i = \pm 1$ for $i = 1, \ldots, N$ such that*

$$w = \prod_{i=1}^{N} p_i r_i^{\epsilon_i} p_i^{-1} \text{ in } F \text{ and } N < f(\ell(w)).$$

It is not hard to see that the existence of a polynomial Dehn function of degree $d \geq 1$ for one finite presentation of the group G implies the existence of a polynomial Dehn function of the same degree d for any other finite presentation (see III.7). Thus the existence of such a Dehn function is independent of the choice of finite presentation of G. Hence we say that G satisfies a *linear, quadratic, cubic, etc. isoperimetric inequality* if it has a finite presentation with a linear, quadratic, cubic, etc., Dehn function. Similar remarks hold also for Dehn functions which are exponential.

The immediate relevance of these notions is that Gromov in his fundamental paper [Gr] has proved the remarkable fact that *a finitely presented group is negatively curved if and only if it satisfies a linear isoperimetric inequality.*

We have already pointed out that these negatively curved groups are also automatic. They include finitely generated free groups, finite groups, cocompact groups of isometries of n-dimensional hyperbolic space and various classes of small cancellation groups (see Gersten and Short [GS1], [GS2] for the latest word on this subject). In addition it follows readily from B.B. Newman's spelling theorem [N] that *one-relator groups with torsion are also negatively curved.*

The connection between automatic groups and these notions is that Cannon et al [CEHPT] have proved that *automatic groups, which we have already noted are finitely presented, satisfy a quadratic isoperimetric inequality.* And similarly, *asynchronously automatic groups satisfy an exponential isoperimetric inequality* [CEHPT].

Each word $w = b_1 \ldots b_n \in A^\star$ can be turned into a map from the set of non-negative integers $0, 1, \ldots$ into G by setting $w(t) = \overline{b_1 \ldots b_t}$ $(t \leq n)$, $w(t) = \overline{w}$ $(t > n)$. We then term two words $u, v \in A^\star$ *k-fellow travellers* if for all t, $d(u(t), v(t)) \leq k$ in $\Gamma_{\mathcal{A}}(G)$. We term u, v *asynchronous k-fellow travellers* if we can find monotonic functions ϕ and ψ so that $d(u(\phi(t)), v(\psi(t))) \leq k$ in $\Gamma_{\mathcal{X}}(G)$. Both automatic and asynchronously automatic groups can be characterized using these "k-fellow traveller" notions.

6. Amalgamated products of automatic and asynchronously automatic groups.

Before stating our results we refer the reader to the book by Lyndon and Schupp [LS] for the definitions and exposition of some of the properties of amalgamated products and HNN extensions.

Let X be a group and let Z be a subgroup of X. Suppose that there exists a rational structure $(\mathcal{X}, L(X))$ for X such that $\mu^{-1}(Z) \cap L(X)$ is regular. Then Gersten and Short [GS3] term Z an $L(X)-$ *rational subgroup* or more briefly a *rational subgroup* of G (cf. also Gilman [Gi]). It follows easily from the work of Gilman [Gi] that such rational subgroups are finitely generated (see also Gersten and Short [GS3]). Suppose now that we denote the *set* of right cosets xZ of Z in X by X/Z. We term a regular set $L(X/Z)$ contained in $L(X)$ a *regular language with uniqueness for* X/Z if the mapping

$$w \mapsto \overline{w}Z \ (w \in L(X/Z))$$

is a bijection between $L(X/Z)$ and X/Z.

All of our theorems about amalgamated products will be proved by appealing to the following very general theorem.

Theorem A. *Let G be the generalised free product of the automatic groups X and Y amalgamating Z:*

$$G = X \star_Z Y.$$

Let $\mathcal{X}$ be a finite set of monoid generators for X, let $\mathcal{Y}$ be a finite set of monoid generators for Y, let $(\mathcal{X}, L(X))$ be an automatic structure for X and let $(\mathcal{Y}, L(Y))$ be an automatic structure for Y. Suppose that the following conditions hold for some constant $k > 0$:

(1) *Z is an $L(X)$-rational subgoup of X (and hence there is a regular language $L(Z) \subseteq L(X)$ with exactly one representative for each element of Z);*
(2) *there is a regular language $L(X/Z)$ with uniqueness for X/Z, contained in $L(X)$ and a regular language $L(Y/Z)$ with uniqueness for Y/Z contained in $L(Y)$;*
(3) *whenever $u \in L(Z)$ and $v \in L(Y)$ represent the same element of Z, then u and v are k-fellow travellers in $\Gamma_{\mathcal{X}\cup\mathcal{Y}}(G)$;*
(4) *if $u \in L(X/Z)$, if $v \in L(Z)$ and if $w \in L(X)$ is such that $\overline{uv} = \overline{w}$ then uv and w are k-fellow travellers in $\Gamma_{\mathcal{X}}(X)$; and similarly if $u \in L(Y/Z)$, if $v \in L(Z)$, $w \in L(Y)$ and $\overline{uv} = \overline{w}$ then uv and w are k-fellow travellers in $\Gamma_{\mathcal{X}\cup\mathcal{Y}}(G)$.*

Then G is automatic. If (3) and (4) are replaced by

(3′) *whenever $u \in L(Z)$ and $v \in L(Y)$ represent the same element of Z, then u and v are asynchronous k-fellow travellers in $\Gamma_{\mathcal{X}\cup\mathcal{Y}}(G)$;*
(4′) *if $u \in L(X/Z)$, if $v \in L(Z)$ and $w \in L(X)$ is such that $\overline{uv} = \overline{w}$ then uv and w are asynchronous k-fellow travellers in $\Gamma_{\mathcal{X}}(X)$; and similarly if $u \in L(Y/Z)$, if $v \in L(Z)$, $w \in L(Y)$ and $\overline{uv} = \overline{w}$ then uv and w are asynchronous k-fellow travellers in $\Gamma_{\mathcal{X}\cup\mathcal{Y}}(G)$.*

Then G is asynchronously automatic.

The assertion in (1) that there is a regular language $L(Z)$ for Z with exactly one representative for each element of Z follows from general principles as we shall see in due course.

There is an analogous theorem which holds also when X and Y are asynchronously automatic.

7. Amalgamated products of finitely generated abelian groups.

We noted earlier that a finitely generated abelian group is automatic. Moreover it follows readily from the fact that a subgroup of a finitely generated abelian group is a direct factor of a subgroup of finite index that subgroups of finitely generated abelian groups are rational (for an appropriate choice of alphabet and language). It follows that we have verified that at least some of the hypothesis of Theorem A is in effect when X and Y are finitely generated abelian groups. Indeed it is not hard to deduce the following theorem from Theorem 1.

Theorem B. *Every amalgamated product of two finitely generated abelian groups is automatic.*

So it follows, in particular, that the groups

$$G_{m,n} =< a, b; a^m = b^n >$$

(and hence also all torus knot groups) are automatic.

8. Amalgamated products of negatively curved groups.

Suppose now that H is a negatively curved group, that $\mathcal{W}$ is a finite set of group generators of H and that $\mathcal{B} = \mathcal{W} \cup \mathcal{W}^{-1}$. A word $w \in \mathcal{B}$ is termed a *geodesic word* if $\ell(w) \leq \ell(v)$ for all words v over $\mathcal{B}$ satisfying $\overline{v} = \overline{w}$. We have already remarked that such a negatively curved group H is automatic. In fact if we denote, for the moment, the set of all geodesic words w over $\mathcal{B}$ by $L(H)$, then $(\mathcal{B}, L(H))$ is an automatic structure for H (see Theorem 1 of II.B.6). In view of the fact that negatively curved groups are automatic, it makes sense to try to apply Theorem A in the case where X and Y are negatively curved. The outcome in this instance is the following theorem, which is perhaps our main result.

Theorem C. *Let X and Y be negatively curved groups and let*

$$G = X \star_Z Y$$

be an amalgamated product of X and Y amalgamating a subgroup Z. Suppose that we can find a finite set $\mathcal{X}$ of group generators of X and a finite set $\mathcal{Y}$ of group generators of Y such that Z is $L(X)$-rational in X, where $L(X)$ is the set of all geodesic words over $\mathcal{X} \cup \mathcal{X}^{-1}$ and Z is $L(Y)$-rational in Y, where $L(Y)$ is the set of all geodesic words over $\mathcal{Y} \cup \mathcal{Y}^{-1}$. Then G is asynchronously automatic. Suppose, in addition, that the finite generating sets $\mathcal{X}$ and $\mathcal{Y}$ can be chosen so that there is a constant k' such that every pair of words $u \in L(X)$, $v \in L(Y)$ which represent the same element $z \in Z$ are k'-fellow travellers in the Cayley graph of G relative to the set $\mathcal{X} \cup \mathcal{Y}$ of group generators. Then G is automatic.

9. Amalgamated products with cyclic amalgamations.

Theorem C can be improved upon when Z is cyclic.

Theorem D. *An amalgamated product of two negatively curved groups with a cyclic subgroup amalgamated is automatic.*

Now finitely generated free groups are negatively curved. So Theorem C applies also to amalgamated products of finitely generated free groups with a cyclic subgroup amalgamated. Negatively curved groups do not contain free abelian subgroups of rank two [Gr]. On the other hand each torus knot group contains a free abelian group of rank two. So it follows that the free product of two negatively curved groups with a cyclic amalgamation need not be negatively curved. On the other hand Gromov asserts on page 113 of [**Gr**], §3.3 that if in addition both factors are torsion free and the cyclic subgroup is maximal in both of them, then the resulting product is negatively curved (see also [BF]).

10. Amalgamated products of free groups.

There is a special case of Theorem C that is of independent interest.

Theorem E. *Every amalgamated product of two finitely generated free groups with a finitely generated subgroup amalgamated is aysnchronous-ly automatic.*

Here Theorem C applies immediately on noting, by a theorem of Anissimov and Seifert [AS] that a finitely generated subgroup of a finitely generated free group is rational. The rationality of finitely generated subgroups of finitely generated free groups is also a consequence of the fact that a finitely generated subgroup of a finitely generated free group is a free factor of a subgroup of finite index (M. Hall [H]) . As we pointed out a little earlier, asynchronously automatic groups are finitely presented and have solvable word problem. So it follows, in particular, that the amalgamated products in Theorem E all have solvable word problem (this can, of course, be proved directly). On the other hand there exist asynchronously automatic groups with unsolvable conjugacy problem because C.F. Miller III [**M**] has proved that there are free products of two finitely generated free groups with a finitely generated subgroup amalgamated which have unsolvable conjugacy problem. It is as yet unresolved whether there exist automatic groups with unsolvable conjugacy problem, although the conjugacy problem is solvable for negatively curved groups (Gromov [Gr]).

Now C.F. Miller III [M] has also proved that the isomorphism problem for free products of finitely generated free groups with a finitely generated subgroup amalgamated is unsolvable. It follows that the isomorphism problem for asynchronously automatic groups is unsolvable. Again the corresponding problem for automatic groups is still open and considered to be of major importance.

Theorem E can be applied also to HNN extensions (see Lyndon and Schupp [LS] for the appropriate notions involved here).

Corollary E1. *Let G be an HNN extension of a finitely generated free group with finitely many stable letters. If the associated subgroups are all finitely generated, then G is asynchronously automatic.*

This corollary follows from Theorem E on noting that G is a free factor of an amalgamated product of two finitely generated free groups with a finitely generated subgroup amalgamated and then appealing to the following

Theorem F. *A free product of two groups is asynchronously automatic (automatic, negatively curved) if and only if the factors are asynchronously automatic (automatic, negatively curved).*

As we have already pointed out, the if part of this proposition is due to Cannon et al [CEHPT]. Somewhat surprisingly the proof of the other half is not completely obvious (see also [GS3]). We shall discuss and prove some variations on Theorem E in III.6. Its counterpart for direct products is unresolved.

11. Examples.

Asynchronously automatic groups which are not automatic are hard to come by. We shall show that the group

$$G = \langle a, b, t, u; tat^{-1} = ab, tbt^{-1} = a, uau^{-1} = ab, ubu^{-1} = a \rangle$$

does not satisfy a quadratic isoperimetric inequality and is consequently not automatic. However G is an HNN extension with two stable letters of the free group of rank two. As we have already pointed out such HNN extensions are all asynchronously automatic.

Now G is a free factor of an amalgamated product H of two finitely generated free groups with a finitely generated subgroup amalgamated. So it follows from Theorem F that there exist amalgamated products of two finitely generated free groups with a finitely generated subgroup amalgamated which are not automatic. In other words Theorem E is best possible. In a vague sense this example touches on the difficulty in obtaining information about the subgroup structure of automatic groups about which we know very little. Recently our knowledge has increased somewhat because of the work of Gersten and Short [GS3]. We also refer the reader to Gersten [Ge] for a detailed discussion of isoperimetric inequalities and for additional examples of asynchronously automatic groups which are not automatic.

Part II. An introduction to automatic groups.

A. Automata

1. Finite state automata and regular sets.

Our objective here is to provide the reader with an introduction to some standard material in Computer Science, most of which can be found, e.g. in Hopcroft and Ullman [HU].

As usual a non-empty set equipped with a binary associative multiplication is termed a *monoid* if it has an identity element.

Let $\mathcal{A}$ be the finite set $\{a_1, \ldots, a_q\}$ and let $\mathcal{A}^\star$ be the set of all *words*

$$w = b_1 \ldots b_n$$

made up from *letters* b_i in $\mathcal{A}$ (including the empty word e); we term n *the length of* w, which we denote by $\ell(w)$. This set $\mathcal{A}^\star$ together with the binary operation concatenation (i.e. juxtaposition) is a monoid, which we will make use of throughout. Let α be the obvious map of $\mathcal{A}$ into $\mathcal{A}^\star$. Then for every monoid E and every map β of $\mathcal{A}$ into E there is a unique monoid homomorphism γ (which by definition maps e to the identity element of E) from $\mathcal{A}^\star$ into E such that $\gamma\alpha = \beta$, i.e. $\mathcal{A}^\star$ is *free* on $\mathcal{A}$. The subsets of $\mathcal{A}^\star$ are often referred to as *languages over* $\mathcal{A}$. We shall be concerned with special languages over $\mathcal{A}$ termed *regular languages over* $\mathcal{A}$ or more usually *regular sets (over* $\mathcal{A}$*)*. These depend for their definition on the notion of a finite state automaton.

Definition. *A finite state automaton is a quintuple*

$$M = (S, Y, A, \tau, s_0),$$

where

(1) *S is a finite set of states;*
(2) *Y is a subset of S, the set of yes states or accept states or final states;*
(3) *$\mathcal{A}$ is a finite set, the alphabet, the elements of which are letters;*
(4) *τ is a function from $S \times \mathcal{A} \to S$, the transition function;*
(5) *s_0 is an element of S, the initial state or start state.*

M can be thought of as a machine with a head that scans a tape, which is in a vertical position and is fed into M. The tape is divided up into a finite number of squares. Each square has a letter printed on it. The top of the tape is fed into M, which starts up in the initial state s_0. M reads the first letter on the tape, the tape moves up so that the machine now scans the second letter on the tape, whereupon the machine goes into a new state. This new state is determined by the transition function τ and the first letter scanned by M. The process continues with each new state being determined by the preceding state and the letter that was scanned while the machine was in that state. When the last letter on the tape is read, the machine goes into a new state and stops. If the last state is an accept state, then the string of letters on the tape is *accepted* by M. Otherwise it is *rejected.*

Definition. *The language recognized by M or the language accepted by M or the language of M is the set $L(M)$ of words accepted by M.*

The language $L(M)$ of M can therefore be described as follows. Given a word $w = b_1 \dots b_n$ $(b_i \in \mathcal{A})$, let $t_0 = s_0$ and let $t_i = \tau(t_{i-1}, b_i)$ $(i = 1, \dots, n)$. Then

$$L(M) = \{w = b_1 \dots b_n \mid t_n \in Y\}.$$

Definition. *A language recognised by a finite state automaton is termed a regular language or a regular set.*

Example.

Let M be the finite state automaton defined as follows. The set S of states of M consists of s_0, the initial state and s_1, s_2, f. The set Y of accept states of M consists of s_0, s_1, s_2, the alphabet $\mathcal{A}$ of M consists of x, X and the transition function τ of M is defined as follows:

$$\tau(s_0, x) = s_1, \tau(s_0, X) = s_2, \tau(s_1, x) = s_1,$$
$$\tau(s_1, X) = f, \tau(s_2, x) = f, \tau(s_2, X) = s_2.$$

Then it is not hard to see that the language $L(M)$ of M consists of all those words in x, X which do not contain a pair of consecutive letters of either the form xX or Xx.

There is a slightly more precise way of defining regular sets over an alphabet $\mathcal{A}$. To this end, let S be a finite set and let $Map(S, S)$ denote the monoid of all maps from S into itself. An *action* of $\mathcal{A}$ on S is by definition a mapping

$$\tau : \mathcal{A} \longrightarrow Map(S, S).$$

Since $\mathcal{A}^\star$ is free on $\mathcal{A}$ this action τ can be extended to a homomorphism $\tau^\star$ of $\mathcal{A}^\star$ into $Map(S, S)$. We sometimes denote the image of $s \in S$ under the mapping $\tau^\star(w)$, where $w \in \mathcal{A}^\star$, by sw. Notice that every finite state automaton comes equipped with such an action on a finite set. It follows immediately that a subset L of $\mathcal{A}^\star$ is regular if we can find an action τ of $A^\star$ on a finite set S, a subset Y of S and an element s_0 of S such that

$$L = \{w \in \mathcal{A}^\star \mid s_0 w \in Y\}.$$

In working with finite state automata we will often suppress explicit mention of τ, using the notation sw introduced above without further explanation.

2. State graphs or transition diagrams.

It is often more convenient to codify a finite state automaton M as a finite *directed , labelled, graph* with a *distinguished vertex* termed the *state graph of M* or the *transition diagram of M.* The *vertices* of this graph are the states of M and the *edges* of M are the triples (s, a, t), where $s \in S$, $a \in A$ and $t = \tau(s, a)$; sometimes a is referred to as a *transition (from s to t).* The *origin* of (s, a, t) is s, the *terminus* is $t = \tau(s, a)$ and the *label* is a. The disinguished vertex of the graph is s_0. Clearly distinct edges can well have the same label, though not if they have the same origin. The sequence of labels of a path in this graph is a word $w \in A^\star$. It follows that the language of M is the set of such words w corresponding to the paths in the graph which begin at s_0 and end at an element of Y. It is useful to term a state *live* if there is at least one path from this state to some accept state; otherwise it is termed *dead*. We term a state f a *fail state* if it is not an accept state and every edge with origin f has terminus f.

Conversely a finite, directed graph Γ, with a distinguished vertex, whose edges are labelled, and which satisfies the obvious conditions, can be thought of as a finite state automaton. More precisely, let S denote the set of all vertices of Γ, $\mathcal{A}$ the set of all labels and s_0 the distinguished vertex and let Y be a subset of S. Suppose that for each vertex s and each label a, there is exactly one edge in Γ with origin s and label a. Then we can define a map τ from $S \times \mathcal{A}$ into S by setting $\tau(s, a) = b$ where b is the terminus of the edge in Γ with origin s and label a.

3. Non-deterministic finite state automata.

We will also need the notion of a *non-deterministic, finite state automaton*, which was introduced by Rabin and Scott [**RS**]. The definition is analogous to that of a finite state automaton.

Definition. *A non-deterministic finite state automaton is a quintuple*

$$M = (S, Y, A \cup \{\epsilon\}, \tau, S_0),$$

where

(1) *S is a finite set of states;*
(2) *Y is a subset of S, the set of accept states or final states;*
(3) *$\mathcal{A}$ is a finite set, the alphabet, the elements of which are letters and ϵ is a special letter not in $\mathcal{A}$;*
(4) *τ is a function from $S \times (\mathcal{A} \cup \{\epsilon\}) \to 2^S$, the transition function, where 2^S is the set of all subsets of S (in the event that $\tau(s, x) = \emptyset$, we sometimes say $\tau(s, x)$ is undefined);*
(5) *S_0 is a non-empty subset of S, the set of initial or start states of M.*

As before M can be thought of as a machine with a head that scans a tape, which is in a vertical position. The tape is divided up into a finite number of squares. Each square has either a letter from $\mathcal{A}$ or an ϵ printed on it. The top of the tape is fed into M, which starts up in any one of the initial states $s \in S_0$. M reads the first letter say x on the tape, and then moves on to the second letter and the machine goes into a new state. This new state is non-deterministic in that it is allowed to be any one of the states in the set $\tau(s, x)$. If $\tau(s, x)$ is undefined, the machine stops. Otherwise the process continues until the last letter on the tape is read. The machine then either stops because the image under the transition function is empty, or else the machine goes into a new state (again as dictated by the transition function) and stops. The transition function τ allows for a certain amount of choice or indeterminacy. If these choices can be made in such a way that the entire word w written on the tape is "read" by M and if the machine goes into an accept state after reading the last letter of w then w is *accepted* by M; otherwise it is *rejected*.

Definition. *The language of the non-deterministic, finite state automaton M is the set $L(M)$ of words accepted by M, with all occurrences of ϵ deleted.*

We emphasise here that the language of a non-deterministic finite state automaton does not consist of the words accepted by the automaton, but consists instead of the words left over after deleting all occurrences of ϵ from such accepted words.

The state graph or transition diagram of a non-deterministic finite state automaton is defined in much the same way as for a finite state automaton. The difference here is that different edges with the same state as origin may well have the same label.

The following theorem of Rabin and Scott [**RS**] shows that the languages recognized by non-deterministic finite state automata are precisely the languages recognized by (deterministic) finite state automata.

Theorem 1. *The language recognised by a non-deterministic finite state automaton is a regular language.*

We will frequently make use of this theorem in order to prove that various languages are regular.

Proof. Let M be a non-deterministic finite state automaton. We first define a new non-deterministic finite state automaton M' as follows. The alphabet, set of states, initial states, accept states of M' are exactly the same as those of M. The transition function τ' of M' is defined by setting, for each $a \in A$,

$$\begin{aligned}\tau'(s,a) =& \tau(s,a) \cup \{u \mid \text{there exist distinct elements } t = t_0, \dots, t_k \text{ of } S \\ & \text{such that } t_0 \in \tau(s,a), t_k = u \text{ and } \tau(t_i, \epsilon) = t_{i+1}, \\ & i = 0, \dots, k-1\}\end{aligned}$$

and

$$\tau'(s,\epsilon) = \emptyset.$$

It follows that the language of M' coincides with the language of M. However we have arranged that if a word w is accepted by M and if w contains any epsilons then the word w' obtained from w by deleting all occurrences of ϵ is accepted by M'. Now we build a finite state automaton whose states consist of the set of all subsets of the set of states of M'. We take the initial state of our finite state automaton to be the set of initial states of M' and the accept states to be those subsets containing an accept state of M'. The transition function σ is defined in the obvious way:

$$\sigma(U,a) = \{t \mid t \in \tau'(u,a), \ \textit{where} \ u \in U\}.$$

It follows without difficulty that the language of this finite state automaton coincides with that of M' and therefore also with the language of M.

4. Calculus of regular sets.

Our objective now is to prove that a variety of sets are regular and that new regular sets can be produced from old by the standard operations of set theory and also by other means. To this end let $\mathcal{A}$ be a finite set and let K and L be subsets of $\mathcal{A}^\star$. We define

$$KL = \{w \mid w = uv, u \in K, v \in L\},$$

and, letting e denote the empty string as usual,

$$K^\star = \{e\} \cup K \cup KK \cup KKK \cup \dots .$$

Theorem 1. *Suppose that K and L are regular sets contained in $\mathcal{A}^\star$. Then the following hold.*

(1) *A finite subset of $\mathcal{A}^\star$ is regular.*
(2) *$\mathcal{A}^\star - K$ is regular.*
(3) *$K \cup L$ is regular.*
(4) *$K \cap L$ is regular.*
(5) *KL is regular.*
(6) *$K^\star$ is regular.*
(7) *$\mathcal{A}^\star$ and $\emptyset$ are regular.*
(8) *If $\mathcal{B}$ is a second finite set and ϕ is a homomorphism of the monoid $\mathcal{A}^\star$ into the monoid $\mathcal{B}^\star$, then $\phi(L)$ is regular over $\mathcal{B}$.*
(9) *If ϕ is a homomorphism of $\mathcal{A}^\star$ into $\mathcal{B}^\star$ and if J is a regular subset of $\mathcal{B}^\star$, then $\phi^{-1}(J)$ is regular over $\mathcal{A}$.*

Proof. (1) Let F be a finite subset of $A^\star$. We define a finite state automaton M with language F as follows. The set S of states of M consists of all the initial subwords of the words in F together with the start state $s_0 = e$ and a *fail* state f. The transition function τ is then defined by $\tau(s, a) = sa$ if $sa \in S$ and $\tau(s, a) = f$ otherwise. The set Y of accept states of M is taken to be F.

(2) If $M = (S, Y, A, \tau, s_0)$ is a finite state automaton recognising the language K, then $(S, S - Y, A, \tau, s_0)$ recognises the language $A^\star - K$.

(3) Let Ψ, Φ be the state graphs corresponding to the finite state automata recognizing the languages K and L respectively. Then the disjoint union of Ψ and Φ is the graph of a non-deterministic automaton (with two initial states), with language $K \cup L$. Since we have made no mention here of ϵ we adopt the convention that all ϵ-transitions map every state to $\emptyset$.

(4) $K \cap L = \mathcal{A}^\star - ((\mathcal{A}^\star - K) \cup (\mathcal{A}^\star - L))$.

(5) Let M be a finite state automaton with $L(M) = K$ and N a finite state automaton with $L(N) = L$, both over the alphabet $\mathcal{A}$. We define now a non-deterministic finite state automaton

$$O = (S \cup T, Z, A \cup \{\epsilon\}, \rho, s_0)$$

where S and T are the set of states of M and N respectively, s_0 is the start state of M, Z is the set of accept states of N and the transition function ρ is defined as follows. On $S \times A$, ρ is the transition function of M and similarly, on $T \times A$, ρ is the transition function of N. The only ϵ-transitions that take a state to something other than $\emptyset$ are those which by definition take an accept state of M to t_0, the initial state of N. Then the language of O is KL.

(6) Suppose that $K = L(M)$, where $M = (S, Y, A, \tau, s_0)$. We define a non-deterministic automaton

$$O = (S \cup \{u_0\}, Y \cup \{u_0\}, A \cup \{\epsilon\}, \sigma, u_0),$$

as follows. As indicated we have introduced a new start state u_0 to M and have enlarged the set of accept states of M by including u_0. The transition function σ is defined τ on $S \times \mathcal{A}$ and $\sigma(u_0, \epsilon) = s_0, \sigma(y, \epsilon) = u_0,\ y \in Y$. As indicated previously, wherever σ has been left undefined, the result is taken to be $\emptyset$. It follows that the language of O is $K^\star$.

(7) The assertions follow from (1), (6) and (2).

(8) Suppose that M is a finite state automaton over $\mathcal{A}$ with language L and that $\mathcal{A} = \{a_1, \ldots, a_q\}$. For each a_i suppose $\phi(a_i) = b_{i,1} \ldots b_{i,m}$, where, of course, m depends on i and each $b_{i,j} \in \mathcal{B}$. For each $s \in S$ we now add new states

$$s_{i,1}, s_{i,2}, \ldots, s_{i,m-1}$$

and define a transition function σ by

$$\sigma(s, b_{i,1}) = s_{i,1}, \sigma(s_{i,1}, b_{i,2}) = s_{i,2}$$
$$, \ldots, \sigma(s_{i,m-2}, b_{i,m-1}) = s_{i,m-1}, \sigma(s_{i,m-1}, b_{i,m}) = \tau(s, a_i),$$

where τ is the transition function of M. This then defines a non-deterministic finite state automaton M' over $\mathcal{B}$, whose accept states are simply the accept states of M, whose start state is the start state of M and whose language is $\phi(L)$.

(9) Let M' be a finite state automaton over $\mathcal{B}$ with $L(M') = J$. We define a finite state automaton M over $\mathcal{A}$ as follows: the states of M, the accept states of M and the start state of M are exactly the same as those of M'. Finally, the transition function τ of M is defined by $\tau(s, a_i) = \sigma(s, \phi(a_i))$, where σ is the transition function of M' and $\sigma(s, \phi(a_i))$ is simply the action of $\phi(a_i)$ on s that is defined by σ. The language of M' is $\phi^{-1}(J)$, which completes the proof of (9).

In general it is not easy to determine whether a given subset of $\mathcal{A}^\star$ is regular. The following lemma can often be used to prove that certain sets are not regular; it is also often useful in proving that a given regular set has some property or other.

Lemma 1 (The Pumping Lemma). *Let L be a regular language over $\mathcal{A}$ and let M be a finite state automaton with k states that recognises L. Then for every word $w \in L$ of length at least k there exists a factorisation*

$$w = xyz$$

of w such that

$$\ell(xy) \leq k, \ell(y) \geq 1,$$

and

$$xy^iz \in L \text{ for every } i \geq 0.$$

Proof. Let us write

$$w = b_1 \ldots b_n (b_i \in \mathcal{A}).$$

The path beginning at the start state s_0 of M in the state graph of M defined by w must contain a loop since $n \geq k$ and there are at most k states in M. Let γ be the first such loop that occurs. We can assume that γ is a simple loop and hence that the subword y of w which "labels" this simple loop is of length at most k. If we think of $\mathcal{A}^\star$ as acting on the set of states of M, as described in §1, then we see that xyz takes s_0 to an accept state of M. It follows that xy^ix takes s_0 to precisely the same state for every $i \geq 0$. Thus $xy^iz \in L$ for every $i \geq 0$.

Notice that the proof of the Pumping Lemma provides us with a little more information which turns out to be useful in some instances. In fact we have proved that if $w' = u'wv' \in L$, where $\ell(w) \geq k$, then we can express w in the form $w = xyz$ with $\ell(xy) \leq k, \ell(y) \geq 1$ so that $u'xy^izv' \in L$ for every $i \geq 0$. We shall often make use of this observation and which we refer to also as the Pumping Lemma.

Corollary 1. *Let M be a finite state automaton. Then there is an algorithm which decides whether or not $L = L(M)$ is finite.*

Proof. Let k be the number of states of M. Notice that by the Pumping Lemma if $w \in L$ and $\ell(w) \geq k$, then there is a word w', satisfying the condition $\ell(w) - k \leq \ell(w') < \ell(w)$, which is also accepted by M. It follows that L is finite if and only if none of the words over $\mathcal{A}$ of length $k, k+1, \ldots, 2k-1$ is accepted by M. We can check this by feeding into M these finitely many words.

5. Products.

Let $\mathcal{A}$ be the usual alphabet. We recall from I.2 that

$$\mathcal{A}(2,\$) = (\mathcal{A} \cup \{\$\}) \times (\mathcal{A} \cup \{\$\}) - \{(\$,\$)\},$$

where $\$$ is the padding symbol introduced in §2. We define similarly

$$\mathcal{A}(3,\$) = (\mathcal{A} \cup \{\$\}) \times (\mathcal{A} \cup \{\$\}) \times (\mathcal{A} \cup \{\$\}) - \{(\$,\$,\$)\}$$

and similarly for $\mathcal{A}(4,\$)$ and so on. The padding map ν also has a counterpart here - if $u, v, w \in \mathcal{A}^\star$, then we define $\nu((u,v,w))$ to be the obvious word in $\mathcal{A}(3,\$)^\star$. So, for example, $\nu(ab, a, aab) = (a,a,a)(b,\$,a)(\$,\$,b)$. Again ν is monic.

We shall make frequent use of the following lemma.

Lemma 1. *If K and L are regular languages over $\mathcal{A}$, then the image of the product set $K \times L$ under ν is a regular language over $\mathcal{A}(2,\$)$.*

Proof. The proof of this proposition is straightforward. We define the appropriate finite state automaton with language $\nu(K \times L)$. In order to do so we need to introduce a new letter q which lies outside $S \cup T$ where (S, Y, A, τ, s_0) and (T, Z, A, σ, t_0) are finite state automata recognising the languages K and L respectively. Consider the finite state automaton with state set $(S \cup \{q\}) \times (T \cup \{q\})$, accept states $(Y \cup \{q\}) \times (Z \cup \{q\}) - \{(q,q)\}$, and alphabet $\mathcal{A}(2,\$)$. The state (q,q) is a fail state and the transition function ς of this finite state automaton takes the value (q,q) except in the following cases:

$$\begin{aligned}
\varsigma((s,t),(a,b)) &= (\tau(s,a), \sigma(t,b)),\ \ s \in S,\ \ t \in T,\ \ a, b \in \mathcal{A} \\
\varsigma((s,t),(\$,b)) &= (q, \sigma(t,b)),\ \ s \in Y \cup \{q\}, t \in T,\ \ b \in \mathcal{A} \\
\varsigma((s,t),(a,\$)) &= (\tau(s,a), q),\ \ s \in S, t \in Z \cup \{q\}, a \in \mathcal{A}.
\end{aligned}$$

The initial state is (s_0, t_0). The language accepted by this finite state automaton is then $\nu(K \times L)$.

Next we prove

Lemma 2. *Suppose that K is regular over the alphabet $\mathcal{A}(2,\$)$. Then the projection P of $\nu^{-1}(K)$ onto the first factor of $\mathcal{A}^\star \times \mathcal{A}^\star$ is regular over $\mathcal{A}$.*

Proof. Let Δ be the state graph of a finite state automaton D with $L(D) = K$. The vertices of Δ are states and the edges of Δ are labelled by pairs (x,y), which lie in $\mathcal{A}(2,\$)$. Let Ψ be the graph with the same set of vertices as Δ. The edges of Ψ are the edges of Δ differently labelled - if (x,y) is the label of an edge in Δ, then this edge in Ψ is labelled by x if $x \neq \$$ and it is labelled by ϵ if $x = \$$. Then Ψ can be thought of as the state graph of a non-deterministic finite state automaton E, whose initial state is the initial state of D and whose accept states are the accept states of D. The language of E is P. So P is regular, as claimed.

There are analogues of Lemma 2 for regular sets over the alphabets $\mathcal{A}(3,\$)$, $\mathcal{A}(4,\$)$ and so on. In these instances there are a number of different possibilities for the projections. However in the cases where we need to make use of such analogues the proof that the sets involved are again regular will closely follow that of Lemma 2 and so we omit it here.

We impose now a total order $\leq$ on $\mathcal{A}$ as follows:

$$a_1 \leq a_2 \leq \cdots \leq a_q.$$

This induces an order, which we again denote by $\leq$ on $\mathcal{A}^\star$, first by length, shorter words coming first and then lexicographically on words of equal length. We will need the following lemmas.

Lemma 3. *The set*

$$L = \{\nu((u,v)) \mid u,v \in \mathcal{A}^\star, \ell(u) < \ell(v)\}$$

is a regular language.

Proof. We construct a finite state automaton M over $\mathcal{A}(2,\$)$ which recognises L. M has three states, s_0, y, f, where s_0 is the start state, y is the only accept state and f is a fail state. The transition function τ of M is defined as follows :

$$\begin{aligned}\tau(s_0,(a,b)) &= s_0, \ \text{ if } a,b \in \mathcal{A};\\ \tau(s_0,(\$,b)) &= y \ \text{ if } b \in \mathcal{A};\\ \tau(y,(\$,b)) &= y \text{ if } b \in \mathcal{A};\\ \tau(s,(c,d)) &= f \text{ in all other cases.}\end{aligned}$$

Next we observe

Lemma 4. *The set*

$$K = \{\nu(u,v) \mid u,v \in \mathcal{A}^\star, \ell(u) = \ell(v), u \leq v\}$$

is regular.

Proof. Let

$$\mathcal{C} = \{(x,x) \mid x \in \mathcal{A}\}, \mathcal{D} = \{(x,y) \mid x,y \in \mathcal{A}\}.$$

Then

$$K = \bigcup_{1 \leq i < j \leq q} (\mathcal{C}^\star(a_i,a_j)\mathcal{D}^\star) \bigcup \mathcal{C}^\star,$$

and is consequently regular.

Proposition 1. *Let Q be a regular set contained in $R = \nu(\mathcal{A}^\star \times \mathcal{A}^\star)$. Then*

$$J = \{u \mid \text{there is at least one } v \text{ with } \nu(u,v) \in Q, \\ \text{and whenever } \nu(u,w) \in Q, \quad \text{then } u \leq w\}$$

is a regular set over $\mathcal{A}$.

Proof. We make use of the notation of Lemma 3 and Lemma 4 of this section. Let π be the composition of ν^{-1}, restricted to R and the projection of $\mathcal{A}^\star \times \mathcal{A}^\star$ onto the first factor. Then

$$J = \pi(Q \cap (K \cup L)) - \pi(Q \cap (R - (K \cup L)))$$

and is consequently regular.

Finally we will need one more fact involving the construction of regular sets.

Lemma 5. *Let L be a regular language over the alphabet $\mathcal{A}$. Then the diagonal*

$$\Delta(L) = \{\nu(w,w) \mid \ w \in L\}$$

of L is regular over $\mathcal{A}(2,\$)$.

Proof. Notice that

$$\Delta(L) = \nu(L \times L) \cap \{(a_i, a_i) \mid i = 1, \ldots, q\}^\star$$

is the intersection of regular sets so it is regular.

6. Asynchronous or two-tape automata.

Definition. *An asynchronous finite state automaton or a two tape automaton T is a deterministic finite state automaton of the form*

$$M = (S, Y, \mathcal{A} \cup \{\epsilon\}, \tau, s_0),$$

where here ϵ is a letter not in $\mathcal{A}$, equipped with a partition

$$S = S_1 \cup S_2$$

of S into two subsets.

We emphasise that the letter ϵ introduced here should not be confused with the ϵ involved in non-deterministic automata, but is used as "an end of tape symbol". We term M *the finite state automaton associated* to T, S the set of *states* of T, s_0 the *start state* of T and Y the set of *accept states* of T.

Now let $w = b_1 \ldots b_n \in (\mathcal{A} \cup \{\epsilon\})^\star$. Put

$$s_i = s_0 b_1 \ldots b_i \ (i = 0, \ldots, n).$$

We now define a mapping

$$\Phi : (\mathcal{A} \cup \{\epsilon\})^\star \longrightarrow (\mathcal{A} \cup \{\epsilon\})^\star \times (\mathcal{A} \cup \{\epsilon\})^\star$$

as follows: $\Phi(w) = (u, v)$ where u is obtained from w by deleting all the letters b_i for which $s_{i-1} \in S_2$ and v is obtained from w by deleting all the letters b_i for which $s_{i-1} \in S_1$. We can think of Φ as a mechanism for taking w and rewriting it on a pair of tapes so that it can now be read by T. Notice that Φ is one-to-one but not in general onto.

Definition. *The language $L(T)$ recognised or accepted by a two-tape automaton T is the set of all pairs $(u', v') \in \mathcal{A}^\star \times \mathcal{A}^\star$ such that*

$$(u'\epsilon, v'\epsilon) \in \Phi(L)$$

where L is the language recognized by the finite state automaton M associated to T. We shall refer to such sets as asynchronously regular sets.

Thus $(u', v') \in L(T)$ if and only if there is a "shuffle" w of $u'\epsilon, v'\epsilon$ which lies in L - so $\Phi(w) = (u'\epsilon, v'\epsilon)$. Note that since M is deterministic, this shuffle, if it exists, is unique.

We can think of T as a machine which scans a pair of input tapes T_1 and T_2. As in the case of a finite state automaton, each of the tapes is divided up into squares. T_1 has a word u' over $\mathcal{A}$ printed on it followed by the *end of tape symbol* ϵ and similarly T_2 has a word v' over $\mathcal{A}$ printed on it followed by an ϵ. The machine starts up in state s_0. If $s_0 \in S_1$, T scans the first letter b_1 printed on T_1 and goes into state $s_0 b_1 = s_1$. Similarly if $s_0 \in S_2$, T scans the first letter c_1 printed on T_2 and goes into state $s_0 c_1 = s_1$. Now if $s_1 \in S_1$ then T scans the second letter b_2 on T_1 when $s_0 \in S_1$ and goes into state $s_1 b_2$. On the other hand if $s_0 \in S_2$ and $s_1 \in S_1$, then T switches over to T_1, scans the first letter b_1 on T_1 and goes into state $s_1 b_1$. At each stage the state that the machine is in dictates which tape is to be read. If the process terminates with T having read everything on both tapes and if the final state that T ends up in is an accept state, then $(u', v') \in L(T)$ - otherwise $(u', v') \notin L(T)$. We sometimes say that $(u'\epsilon, v'\epsilon)$ is recognised or accepted by T. Notice however that it is the pair (u', v') that is in $L(T)$.

Examples.

(1) The set

$$L = \{\nu((a^n, a^{2n})) \mid n = 0, 1, \dots\}$$

is not the language of a finite state automaton over $\mathcal{A}(2, \$)$, where here $\mathcal{A} = \{a\}$.

The proof that L is not the language of a finite state automaton follows readily from the Pumping Lemma.

(2) The set

$$L = \{(a^n, a^{2n}) \mid n = 0, 1, \dot\}$$

is the language of a two-tape automaton.

To see this, define a two-tape automaton T by defining its associated finite state automaton M as follows:

$$\mathcal{A} = \{a\}, S_1 = \{s_0\}, S_2 = \{s_1, s_2, s_3, y, f\}, Y = \{y\}.$$

The transition function τ of M is defined then by

$$s_0 a = s_1, s_0 \epsilon = s_3, s_1 a = s_2, s_2 a = s_0, s_3 a = f, s_3 \epsilon = y$$
$$y\epsilon = f, ya = f, fa = f, f\epsilon = f, s_1\epsilon = f, s_2\epsilon = f.$$

(3) The set

$$L = \{(a^n, a^{2n}) \mid n = 0, 1, \dots\} \cup \{(a^{2n}, a^n) \mid n = 0, 1, \dots\}$$

is not the language of a two tape automaton.

In order to see that L is not asynchronously regular, observe that if an asynchronous machine T is scanning a pair of tapes with $(a^r\epsilon, a^s\epsilon)$ written on them, where either $r = 2n$ and $s = n$ or vice-verse, then for large n there is no way that T can be concocted so as to be able to discern whether to read the first tape twice as fast as the second or the second twice as fast as the first. This remark can be translated into a proof by applying the Pumping Lemma in an appropriate fashion.

It follows that the union of two asynchronously regular sets need not be asynchronously regular. Although we will not prove this here, asynchronously regular sets are closed under complementation. So this implies that the class of asynchronously regular sets is not closed under intersections either.

One could broaden the definition of an asynchronous two–tape automaton to non–deterministic asynchronous two–tape automaton T where the corresponding automaton M may be a non–deterministic finite state automaton. The class of languages of such machines is clearly closed under finite union, and so is a larger class than the class of asynchronously regular sets.

The following theorem serves to show that two tape automata are more powerful than ordinary automata.

Theorem 1. *Let $\mathcal{B}$ be a given alphabet, K a subset of $\mathcal{B}^\star \times \mathcal{B}^\star$ such that $\nu(K)$ is regular over $\mathcal{B}(2,\$)$. Then K is asynchronously regular.*

Proof. Let M be a finite state automaton over $\mathcal{B}(2,\$)$ recognizing $\nu(K)$ and put $\mathcal{A} = \mathcal{B} \cup \{\epsilon\}$. Let

$$S_\ell = \{s_\ell \mid s \in S\},\ S_r = \{s_r \mid s \in S\}$$

be two copies of S. Put

$$S_1 = S \cup S_\ell,\ S_2 = S_r \ \cup\ S \times \mathcal{A}\ \cup \{y\}\ \cup \{f\}.$$

We define now a two-tape automaton T by defining its associated finite state automaton N as follows. First the state set of N is $S_1 \cup S_2$. The start state of N is the start state of M (which lies in S_1), the accept state of N is y and f is a fail state of N. We denote the action of $B(2,\$)$ on S that is given to us by the transition function of M by writing $s(x,y)$, where $s \in S$ and $(x,y) \in B(2,\$)$. Then, with $(b, c \in B, s \in S)$, the transition function of N is defined to take all states to the fail state f except in the instances detailed below:

$$sb = (s,b), (s,b)c = s(b,c), s\epsilon = (s,\epsilon), (s,\epsilon)b = (s(\$,b))_r,$$
$$(s,b)\epsilon = (s(b,\$))_\ell, s_\ell b = (s(b,\$))_\ell, s_r b = (s(\$,b))_r$$
$$s_r\epsilon = y, (s,\epsilon)\epsilon = y \text{ if s an accept state of } M.$$

Then $L(T) = K$.

It follows from Theorem 1 and the very definitions of automatic and asynchronously automatic groups that every automatic group is also asynchronously automatic (see I.2 and I.4).

Recall that the class of non–deterministic asynchronously regular languauges is strictly larger than the class of (deterministic) asynchronously regular languages. However M. Shapiro has observed that the corresponding classes of groups (asynchronously automatic and “non–deterministic asynchonously automatic”) are identical.

7. Asynchronously regular sets.

The class of sets that are the languages of two tape automata is not closed under intersection. However in one important instance the intersection of two asynchronously regular sets is again asynchronously regular.

Lemma 1. *Let L' and L'' be regular sets over the alphabet $\mathcal{A}$ and let L be a language over $\mathcal{A}$ recognized by the two tape automaton T. Then*

$$L \cap (L' \times L'')$$

is the language of a two tape automaton.

Proof. Let M be the finite state automaton associated to T and let M', M'' be finite state automata recognizing the languages L' and L'' respectively. Let $S = S_1 \cup S_2$ be the decomposition of the state set S of M, let S', S'' be the state sets of M' and M'' respectively, with Y, Y', Y'' denoting the respective accept state sets, s_0, s_0' and s_0'' the respective start states. We now define a two tape automaton U whose language is

$$L \cap (L' \times L'').$$

The finite state automaton N associated to U has state set $S \times S' \times S''$, with decomposition

$$S \times S' \times S'' = (S_1 \times S' \times S'') \cup (S_2 \times S' \times S'').$$

The set of accept states of N is $Y \times Y' \times Y''$ and the start state is (s_0, s_0', s_0''). Then the transition function of N is defined for each $a \in \mathcal{A}$ by

$$(s, s', s'')a = \begin{cases} (sa, s'a, s'') \text{ } if \text{ } s \in S_1 \\ (sa, s, s''a) \text{ } if \text{ } s \in S_2 \end{cases}$$

and

$$(s, s', s'')\epsilon = (s\epsilon, s', s'').$$

Then $L(U) = L \cap (L' \times L'')$, as required.

We note finally that it is not hard to see that $\mathcal{A}^\star \times \mathcal{A}^\star$ is asynchronously regular. Hence it follows from Theorem 1 that if K and L are regular over $\mathcal{A}$ then $K \times L$ is asynchronously regular.

B. Automatic and asynchronously automatic groups

1. Regularity and automatic groups.

We will adopt throughout, unless explicitly stated otherwise, the notation used in I.2.

We start out here with some examples of automatic groups.

Example 1. *Finite groups are automatic.*

Proof. Let $G = \{x_1, \ldots, x_q\}$ be a finite group of order q. We take $\mathcal{A} = \{a_1, \ldots, a_q\}$ to be a set of monoid generators for G, the element a_i mapping onto x_i. Put $L = \mathcal{A}$. Then, by Theorem 1 of II.A.4, L is regular and so too are the L_i and $L_=$ since they are all finite.

Example 2. *The infinite cyclic group $C_\infty =< x >$ is automatic.*

Proof. We set $\mathcal{A} = \{x, X\}$ and define μ by putting $\mu(x) = x, \mu(X) = x^{-1}$. Let $L = L_+ \cup L_-$ where

$$L_+ = \{x\}^\star, L_- = \{X\}^\star.$$

Then $L_= = \Delta(L)$,

$$L_x = \Delta(L_+)\{(x, \$)\} \cup \Delta(L_-)\{(\$, X)\}$$

and

$$L_X = \Delta(L_+)\{(\$, x)\} \cup \Delta(L_-)\{(X, \$)\}.$$

It follows then from Theorem 1 of II.A.4, that each of these sets is regular and hence that C_∞ is automatic.

Notice that every element of C_∞ has a unique representative. This can always be arranged by virtue of the following proposition.

Proposition 1. *Let $(\mathcal{A}, L)$ be an automatic structure for the automatic group G. Then there is a regular set J contained in L such that $(\mathcal{A}, J)$ is an automatic structure for G and such that J contains exactly one representative for each element of G.*

Proof. We adopt the notation of Proposition 1 of II.A.5. Recall that we order the finite set $\mathcal{A}$, and then order $\mathcal{A}^\star$ such that $u \leq v$ means that $\ell(u) < \ell(v)$ or $\ell(u) = \ell(v)$ and u precedes v in the induced lexicographical order.

We take here $Q = L_=$. Then the corresponding set J is given by

$$J = \{u \mid u \in L \text{ and if } \nu(u, v) \in L_=, u \leq v\}.$$

So J is regular and consists precisely of those elements $u \in L$ which are lexicographically least among the elements of L which map onto $\mu(u)$ under μ as desired.

It is clear that $(\mathcal{A}, J)$ is a rational structure for G. It remains to prove that it is also an automatic structure. Since $J_= = \Delta(J)$, $J_=$ is regular by Lemma 5 of II.A.5. Similarly J_i is regular since it is the intersection of regular sets:

$$J_i = L_i \cap \nu(J \times J).$$

We call such an automatic structure $(\mathcal{A}, J)$ an *automatic structure with uniqueness.*

Next we prove

Lemma 1. *If $(\mathcal{A}, L)$ is an automatic structure for the group G then, for each $w \in \mathcal{A}^\star$,*

$$L_w = \{\nu(u,v) \mid u, v \in L, \overline{u} = \overline{vw}\}$$

is regular.

Proof. We proceed by induction on $\ell(w)$, the length of w. Suppose that

$$w = b_1 \dots b_n,$$

where here $b_i \in \mathcal{A}$. If $\ell(w) = 0$, i.e. if $n = 0$, then $L_w = L_=$ is regular by hypothesis. Suppose then that $\ell(w) > 0$. Then

$$w = b_1 \dots b_{n-1} b_n = w' b_n.$$

Inductively $L_{w'}$ is regular. Now (note the discussion in the first paragraph of II.A.5 and the remark following the proof of Lemma 2 of that section)

$$L' = \nu(L_{b_n} \times L_{w'}) \cap \nu(L \times \Delta(L) \times L)$$

is regular. Observe that

$$L' = \{\nu(u', u, u, v) \mid \overline{u'} = \overline{ub_n}, \overline{u} = \overline{vw'}\}.$$

We now project L' onto the product of the first and the last factors. This yields the language L_w which is consequently regular by the appropriate analogue of Lemma 2 of II.A.5.

Next we deduce the following theorem of R. Gilman [Gi].

Theorem 1. *Let G be an automatic group. If G is infinite then G contains an element of infinite order.*

Notice that Theorem 1 implies that an automatic group all of whose elements are of finite order is finite.

Proof. Let $(\mathcal{A}, L)$ be an automatic structure with uniqueness for G. Let M be a finite state automaton which recognizes L. Let k be the number of states of M. Since G is infinite, so is L. Let $w \in L$ be of length at least k. By the Pumping Lemma

$$w = xyz \quad \ell(y) \geq 1 \text{ where } xy^iz \in L \ (i \geq 0).$$

Notice that this implies that $\overline{y}$ is of infinite order. Indeed suppose that we assume the contrary. Then the map μ, which is by assumption, monic on L, maps the infinitely many elements xy^iz $(i \geq 0)$ to finitely many elements of G.

The next lemma turns out to be very useful in working with automatic groups.

Lemma 2. *Let $(\mathcal{A}, L)$ be an automatic structure with uniqueness for the group G. Let $M, M_=, M_1, \dots, M_q$ be finite state automata which recognize respectively the regular sets $L, L_=, L_1, \dots, L_q$. Let k be the maximum number of states in the the finite state automata above. Then for each $j \in \{=, 1, \dots, q\}$ and every pair of words $u, v \in L$ for which $\nu(u, v) \in L_j$,*

$$\mid \ell(u) - \ell(v) \mid < k.$$

The finite state automaton $M_=$ is sometimes termed *an equality check-er for G* and the automata M_i are then termed *comparator automata,*

Proof. Suppose that $\ell(u) = \ell(v) + r$ where $r \geq k$. It follows then from the Pumping Lemma that we can find words x, y, z over $\mathcal{A}(2, \$)$ such that

$$\nu(u, v) = xyz, \ell(x) = \ell(v), \ell(y) \geq 1, xy^i z \in L_j \ (i = 1, 2, \dots).$$

Observe that $x = (b_1, c_1) \dots (b_p, c_p)$, that $y = (b_{p+1}, \$) \dots (b_{p+m}, \$)$ and that $z = (b_{p+m+1}, \$) \dots (b_n, \$)$ where here $u = b_1 \dots b_n, v = c_1 \dots c_p$. It follows that the infinitely many elements

$$b_1 \dots b_p (b_{p+1} \dots b_{p+m})^i b_{p+m+1} \dots b_n \in L (i > 0)$$

represent the same element of G. This contradicts the assumption that $(\mathcal{A}, L)$ is an automatic structure with uniqueness for G.

As an example of manipulation of automatic groups, we use Lemma 2 to prove

Theorem 2. ([CEHPT, §13, Theorem 13.2])
The free product G of two automatic groups G_1 and G_2 is again automatic.

Proof. We remind the reader that we can think of G as a group which is generated by its subgroups G_1 and G_2 and which satisfies two additional conditions, namely

(1) $G_1 \cap G_2 = 1$;
(2) every product $g_1 \dots g_n$ ($g_i \in G_1 \cup G_2 - \{1\}$) which is strictly alternating, i.e. if $g_i \in G_1$, then $g_{i+1} \in G_2$ and vice versa, is different from 1.

Let $(\mathcal{A}_i, L_i)$ be an automatic structure with uniqueness for G_i, $i = 1, 2$. Put $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$. Then $\mathcal{A}$ can be viewed as a set of monoid generators for G in the obvious way. We assume in addition that the empty word e is the representative in L_i of the identity element of the group G_i (that this is always possible is shown at the end of the next section). Define

$$L = L_2 \{(L_1 - \{e\})(L_2 - \{e\})\}^\star L_1.$$

Then L is regular. We claim that $(\mathcal{A}, L)$ is an automatic structure with uniqueness for G.

Observe first that L contains exactly one representative for each element of G. Consequently $L_= = \Delta(L)$ and is consequently also regular.

Now define

$$L' = L_2 \{(L_1 - \{e\})(L_2 - \{e\})\}^\star$$

and

$$L'' = L_1 \{(L_2 - \{e\})(L_1 - \{e\})\}^\star.$$

Then L' and L'' are regular. Now if $x \in \mathcal{A}_1$ then

$$L_x = \Delta(L')(L_1)_x$$

and if $x \in \mathcal{A}_2$, then

$$L_x = \Delta(L'')(L_2)_x.$$

So in both cases L_x is regular. This completes the proof that $(\mathcal{A}, L)$ is an automatic structure with uniqueness for G and hence that G is automatic.

There is a more convenient way of proving that a group is automatic which is perhaps best described by considering its Cayley graph.

2. Geometry of the Cayley graph.

Let $\mathcal{X}$ be a finite set of monoid generators of the group G. We can also view $\mathcal{X}$ as a set of group generators (see I.5) and hence we can adopt here the notation and definitions of that section. As before we define

$$\mathcal{A} = \mathcal{X} \cup \mathcal{X}^{-1}$$

and use the notation $\overline{w}$ for the image in G of the word $w \in \mathcal{A}^\star$ under the usual homomorphism μ. We will sometimes denote the elements of $\mathcal{A}$ simply by $a_1, \dots, a_q$. As already discussed in I.5 we denote the Cayley graph of G relative to this set $\mathcal{X}$ of generators variously by $\Gamma = \Gamma(G) = \Gamma_{\mathcal{X}}(G)$.

We adjoin now to $\mathcal{A}$ the padding symbol \$ and extend μ to a homomorphism of $(\mathcal{A} \cup \{\$\})^\star$ onto G by sending \$ to 1. If

$$w = c_1 \dots c_n \in (\mathcal{A} \cup \{\$\})^\star$$

then we define the *t-th initial segment* or *t-th prefix* w_t of w by

$$w_t = c_1 \dots c_t \text{ if } t \leq n, \ w_t = w \text{ if } t > n.$$

Here we interpret $w_0 = e$, the empty word. Each such word w defines an infinite path ω in Γ by

$$\omega(t) = \overline{w_t}(t \geq 0).$$

Notice that ω has origin 1 and "terminates" at $\overline{w}$.

The following lemma can be proved directly from the definition.

Lemma 1. *Let Γ be the Cayley graph of the group G as defined above. Then the following hold:*

(1) *$d(g,h)=n$ if there exists $u \in \mathcal{A}^\star$ of length n such that $g\overline{u} = h$ and if $v \in \mathcal{A}^\star$ is such that $g\overline{v} = h$ then $\ell(v) \geq n$ $(g, h \in G)$;*
(2) *$d(g,h) = d(fg, fh) = d(1, g^{-1}h)$ $(f, g, h \in G)$;*
(3) *$d(g\overline{u}, h\overline{v}) \leq \ell(u) + d(g,h) + \ell(v)$ $(g, h \in G, u, v \in \mathcal{A}^\star)$;*
(4) *$d(g_1 f g_2, h_1 f h_2) = d(g_1 g_2, h_1 h_2)$ if $f, g_1, g_2, h_1, h_2 \in G$ and $fg_1 = g_1 f, fh_1 = h_1 f$.*

Now suppose that $(\mathcal{X}, L)$ is an automatic structure for the group G. Then, adopting the notation introduced above, the following lemma holds.

Lemma 2. *$(\mathcal{A}, L)$ is an automatic structure for G.*

proof. If $x \in \mathcal{X}$, then

$$L_{x^{-1}} = \{\nu(u,v) \mid \nu(v,u) \in L_x\}.$$

We recall the following definition from I.5.

Definition. *Let u, v be two words over $\mathcal{A} \cup \{\$\}$ and let k be a fixed positive real number. We term u, v uniformly k-close or k-fellow travellers if*

$$d(u(t), v(t)) \leq k \text{ for all } t \geq 0.$$

This implies that the two paths defined by u and v (in the Cayley graph Γ of G relative to $\mathcal{X}$) from 1 to $\overline{u}$ and $\overline{v}$ respectively stay within a distance k of each other at each point t in time. Cannon et. al.[CEHPT] refer to such a number k as a *modulus of continuity* and we shall sometimes avail ourselves of this terminology. We sometimes also say that u and v *fellow travel with constant k.*

We will find the following lemma useful.

Lemma 3. *Let $(\mathcal{B}, L)$ be an automatic structure for the automatic group G. Let ℓ be a positive integer and let*

$$\mathcal{R} = \{\nu(u,v) \mid u, v \in L, d(\overline{u}, \overline{v}) \leq \ell\}.$$

Then there exists a positive real number k such that if $\nu(u,v) \in \mathcal{R}$ then u and v are k-fellow travellers.

Proof. Let $w_1, \ldots, w_j$ be the set of all those elements of $\mathcal{B}^\star$ of length at most ℓ. Then each L_{w_i} is regular and therefore so too is

$$\mathcal{R} = \bigcup_{i=1}^{j} L_{w_i}.$$

Let M be a finite state automaton recognizing the regular set $\mathcal{R}$. Suppose that r is the number of states of M. We claim that $k = 2r + \ell$ fulfills the conditions of the lemma. This is a consequence of the proof of the Pumping Lemma and Lemma 1 of this section. In a little more detail, observe that for each t there is a path of length at most $r - 1$ in the state diagram of M from $\nu(u_t, v_t)$ to an accept state of M. This translates into the existence of two words u_1 and v_1 over $\mathcal{B}$ of length at most $r - 1$ such that

$$u(t)\overline{u_1} = v(t)\overline{v_1 w}$$

where w is a third word over $\mathcal{B}$ of length at most ℓ. It follows from Lemma 1 above that

$$d(u(t), v(t)) < 2r + \ell$$

which is more than enough to complete the proof.

Next we introduce another crucial definition.

Definition. *A language L over the alphabet $\mathcal{X}$ has the k-fellow traveller property if u and v in L are k-fellow travellers whenever $d(\overline{u}, \overline{v}) \leq 1$.*

This leads to the following important characterisation of automatic groups.

Theorem 1. ([CEHPT, §6])
Let $(\mathcal{X}, L)$ be a rational structure for the group G. Then $(\mathcal{X}, L)$ is an automatic structure for G if and only if L has the k-fellow traveller property for some k.

Proof. One part of this theorem follows immediately from Lemma 3 of this section - we need only take $\ell = 1$.

In order to prove the converse let us suppose that $\mathcal{X} = \{x_1, \ldots, x_p\}$ and put $x_0 = e$. We now define for each $i = 0, \ldots, p$ a finite state automaton N_i which recognizes $L_i = L_{x_i}$, which ensures that $(\mathcal{X}, L)$ is an automatic structure for G.

Now denote the ball of radius k in the Cayley graph Γ of G relative to $\mathcal{X}$ with center at 1 by $B_k(1)$:

$$B_k(1) = \{g \mid g \in G, d(1, g) \leq k\}.$$

Notice that $B_k(1)$ is finite.

The state set of N_i is $B_k(1) \cup \{f\}$, where f is a fail state. The start state of N_i is 1, and it has exactly one accept state $\overline{x_i}$. The transition function of N_i is then given by

$$g(x, y) = \begin{cases} \overline{y}^{-1} g \overline{x} & \text{if } \overline{y}^{-1} g \overline{x} \in B_k(1) \\ f & \text{otherwise,} \end{cases}$$

where here $(x, y) \in \mathcal{X}(2, \$)$ and we take $\overline{\$^{\pm}} = 1$. Then it follows that N_i accepts $\nu(u, v)$ $(u, v \in L)$ if and only if $\overline{v}^{-1}\overline{u} = \overline{x_i}$. Hence

$$L_i = \nu(L \times L) \cap L(N_i)$$

is regular, by Theorem 1 of II.A.4 and Lemma 1 of II.A.5, and the proof of the theorem is complete.

We shall sometimes refer to these finite state automata N_i as *abbreviated versions of the standard automata.*

It follows from the proof of this theorem that if we replace some representatives in L by other representatives in in $\mathcal{A}^\star$, only the fellow traveller constant changes. It follows from this that

Corollary 1. *Let $(\mathcal{A}, L)$ be an automatic structure for G. Let S be a finite subset of L, and S' a finite subset of $\mathcal{A}^\star$ such that $\mu((L - S) \cup S') = G$.*
Then $(\mathcal{A}, (L - S) \cup S')$ is an automatic structure for G.

In particular, we can always suppose that the empty word is a representative of an automatic structure, if we are free to alter the fellow traveller constant.

3. Automatic structure and generating sets.

Our objective in this section is to prove that the existence of an automatic structure is independent of the choice of monoid generators.

Theorem 1. *Let G be a group and let $\mathcal{A}$ and $\mathcal{B}$ be finite sets of monoid generators for G. Then G has an automatic structure over $\mathcal{A}$ if and only if it has an automatic structure over $\mathcal{B}$.*

Proof. Suppose that $(\mathcal{A}, L)$ is an automatic structure for G. We divide the proof that G also has an automatic structure over $\mathcal{B}$ into three cases.

We consider first the case where $\mathcal{B}$ contains an element b_0 such that $\overline{b_0} = 1$. Suppose then that $\mathcal{A} = \{a_1, \ldots, a_q\}$ and that $\mathcal{B} = \{b_0, \ldots, b_p\}$. Since $b_0 \in \mathcal{B}$ we can choose $w_1, \ldots, w_q \in \mathcal{B}^\star$ such that

$$\overline{a_i} = \overline{w_i} \text{ and } \ell(w_i) = \ell \ (i = 1, \ldots, q)$$

for some ℓ (padding out the w_i's with b_0's where necessary). Now define a homomorphism $\phi : \mathcal{A}^\star \longrightarrow \mathcal{B}^\star$ by

$$\phi(a_i) = w_i \ (i = 1, \ldots, q).$$

By Theorem 1 of II.A.4 $\phi(L) = L'$ is regular over $\mathcal{B}$. We shall prove that L' has the $k-$fellow traveller property for an appropriate choice of k.

Let us denote the distance function in the Cayley graph $\Gamma_{\mathcal{A}}$ of G relative to $\mathcal{A}$ by $d_{\mathcal{A}}$ and similarly denote the distance function in the Cayley graph $\Gamma_{\mathcal{B}}$ of G relative to $\mathcal{B}$ by $d_{\mathcal{B}}$.

Choose $u_j \in \mathcal{A}^\star$ such that

$$\overline{u_j} = \overline{b_j} \ (j = 1, \ldots, p).$$

Let

$$\ell' = max\{\ell(u_1), \ldots, \ell(u_p)\}.$$

By Lemma 2 of II.B.1 there exists a constant k_1 such that if $d(\overline{u}, \overline{u'}) \leq \ell'$ where $u, u' \in L$, then u and u' are k_1-fellow travellers in $\Gamma_{\mathcal{A}}$. We now choose

$$k = k_1\ell' + 2\ell.$$

Now suppose that $w, w' \in L'$ and that $d_{\mathcal{B}}(\overline{w}, \overline{w'}) \leq 1$. We will prove that w and w' are k-fellow travellers in $\Gamma_{\mathcal{B}}$. By definition, we can express

$$w = w_{i_1} \ldots w_{i_r}, \text{ where } u = a_{i_1} \ldots a_{i_r} \in L$$

and

$$w' = w_{j_1} \ldots w_{j_s}, \text{ where } u' = a_{j_1} \ldots a_{j_s} \in L.$$

Since $\overline{u} = \overline{w}$ and $\overline{u'} = \overline{w'}$, it follows that $d_{\mathcal{A}}(\overline{u}, \overline{u'}) \leq \ell'$. So u and u' are k_1-fellow travellers in $\Gamma_{\mathcal{A}}$. Therefore w and w' are fellow travellers in $\Gamma_{\mathcal{B}}$ with constant $k = k_1\ell' + 2\ell$.

We consider next the case $\mathcal{A} = \{b_0, \ldots, b_p\}$ $(p > 0)$, $\mathcal{B} = \{b_1, \ldots, b_p\}$ where again we assume that $\overline{b_0} = 1$. Let $(\mathcal{A}, L)$ be an automatic structure for G over $\mathcal{A}$. Choose $v \in \mathcal{B}^\star$ with $\overline{v} = 1$ such that $\ell(v) = \ell \geq 1$. Define a set map $\Psi : \mathcal{A}^\star \longrightarrow \mathcal{B}^\star$ as follows. If $w \in \mathcal{A}^\star$, number the occurrences of b_0 consecutively. Then $\Psi(w)$ is the result of omitting from w all occurrences of b_0 numbered by integers which are incongruent to 0 modulo ℓ and replacing the others by v. We now define

$$L' = \Psi(L).$$

We claim that $(\mathcal{B}, L')$ is an automatic structure for G. There are two parts to the proof of this assertion. The first involves proving that L' has the k-fellow traveller property, for some k - our choice of L' was designed with this in mind. The second part is the proof that L' is regular. Since L' is clearly a language for G, this will mean that $(\mathcal{B}, L')$ is indeed an automatic structure for G.

Suppose that k-fellow traveller constant for L is k_1. We shall prove that L' has the k-fellow traveller property with $k = k_1 + 2\ell$. Notice that the distance functions in $\Gamma_{\mathcal{A}}$ and $\Gamma_{\mathcal{B}}$ are the same, as an edge labelled b_0 begins and ends at the same vertex, and so appears in no geodesic path. So we shall denote them both by d. Now if $w \in L$, then for each t we have

$$d(w(t), \Psi(w)(t)) \leq \ell.$$

Suppose that $d(\overline{\Psi(w)}, \overline{\Psi(w')}) \leq 1$ in $\Gamma_{\mathcal{B}}$. Then $d(\overline{w}, \overline{w'}) \leq 1$ in $\Gamma_{\mathcal{A}}$. Consequently w and w' are k_1-fellow travellers in $\Gamma_{\mathcal{A}}$ and hence also k_1-fellow travellers in $\Gamma_{\mathcal{B}}$. It follows that $\Psi(w)$ and $\Psi(w')$ are $k_1 + 2\ell$-fellow travellers in $\Gamma_{\mathcal{B}}$ as desired.

It remains to prove that L' is regular. We build a non-deterministic finite state automaton M' accepting L'. Suppose first that

$$M = (S, Y, \mathcal{A}, \tau, s_0)$$

is a finite state automaton accepting L. We define the state set S' of M' in stages. To begin with S' contains $S \times \{1, \ldots, \ell\}$. The transition function is defined on this part of S' as follows:

$$(s, r)b_i = (sb_i, r) \; (i = 1, \ldots, p).$$

We now define a number of ϵ-transitions for our machine M':

$$(s, r)\epsilon = (sb_0, r + 1) \; (r = 1, \ldots, \ell - 1).$$

We now complete the definition of S'. To this end suppose that $v = c_1 \ldots c_\ell$. Then for each $s \in S$ we add the states (s, ℓ, c_i) $(i = 1, \ldots, \ell - 1)$ and complete the definition of the transition function of M' by defining

$$(s, \ell)c_1 = (s, \ell, c_1), (s, \ell, c_1)c_2 = (s, \ell, c_2), \ldots, (s, \ell, c_{\ell-1})c_\ell = (s, 1).$$

We have one initial state for M', namely $(s_0, 1)$ and take $Y \times \{1, \ldots, \ell\}$ to be the accept states of M'. Then it follows that the language of M' is L' as desired.

We are now in a position to complete the proof of the theorem. Suppose then that $\mathcal{A}$ and $\mathcal{B}$ are any given pair of monoid generators for G, as described in the statement of Theorem 1. We adjoin to $\mathcal{B}$ a new letter b_0 and define $\overline{b_0} = 1$. Then by the first part of the proof G is automatic over $\mathcal{B} \cup \{b_0\}$ and by the second part of the proof it is automatic also over $\mathcal{B}$, as required.

4. Automatic groups.

Our objective now is to prove the following

Theorem 1. *The direct product G of two automatic groups G_1 and G_2 is again automatic.*

Proof. We think of G as being generated by its normal subgoups G_1 and G_2 which intersect in the identity and which commute elementwise. In particular every element $g \in G$ can be written uniquely in the form $g = g_1 g_2$ where $g_i \in G_i$. Now let $(\mathcal{A}_1, L_1)$ and $(\mathcal{A}_2, L_2)$ be automatic structures with uniqueness for G_1 and G_2 respectively. We put $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ and $L = L_1 L_2$. Then $(\mathcal{A}, L)$ is a rational structure with uniqueness for G. It suffices then for the proof of the theorem to show that L has the k-fellow traveller property for some k.

With this in mind, observe that L_i has the k_i-fellow traveller property for some choice of k_i. In addition, by Lemma 2 of II.B.1 there exists a real number k_3 such that if $u_1, v_1 \in L_1$ are such that $d(\overline{u_1}, \overline{v_1}) \leq 1$ then

$$| \ell(u_1) - \ell(v_1) | \leq k_3.$$

Let k_4 be the maximum of k_1, k_2, k_3 and put $k = 2k_4$. We shall prove that L has the fellow traveller property with constant $k = 2k_4$.

Suppose that $u, v \in L$ and that $d(\overline{u}, \overline{v}) \leq 1$. If $d(\overline{u}, \overline{v}) = 0$ then $u = v$ and so are trivially $k-$fellow travellers. Suppose then that $d(\overline{u}, \overline{v}) = 1$. Now

$$u = u_1 u_2, \ v = v_1 v_2 \ (u_1, v_1 \in L_1, u_2, v_2 \in L_2)$$

uniquely and $\overline{u} = \overline{vx}$ for some $x \in \mathcal{A}$. If $x \in \mathcal{A}_2$ then $u_1 = v_1$ and $\overline{u_2} = \overline{v_2 x}$. Hence u_2 and v_2 are $k-$fellow travellers. It follows then from Lemma 2 of II.B.2 that u and v are also k-fellow travellers. Finally, suppose that $x \in \mathcal{A}_1$. Then $\overline{u_1} = \overline{v_1 x}$ and so u_1 and v_1 are k_4-fellow travellers. Moreover $| \ell(u_1) - \ell(v_1) | \leq k_4$. We consider now in turn each of the possibilitites (i) $\ell(u_1) < \ell(v_1)$, (ii) $\ell(u_1) = \ell(v_1)$ and (iii) $\ell(u_1) \geq \ell(v_1)$. It follows from Lemma 1 of II.B.2 that

$$d(u(t), v(t)) \leq (2k_4 =)k$$

for every t. This completes the proof.

Next we prove

Theorem 2. *Let G be a group and let G' be a subgoup of finite index m in G. Then G is automatic if and only if G' is automatic.*

Proof. Suppose first that G' is automatic. Decompose G into m cosets modulo G':

$$G = G' r_1 \cup G' r_2 \cup \cdots \cup G' r_m$$

where we take $r_1 = 1$. Let $(\mathcal{A}', L')$ be an automatic structure with uniqueness for G', where $\mathcal{A}' = \{b_1, \ldots, b_p\}$. Put

$$\mathcal{A} = \{b_1, \ldots, b_p, c_1, \ldots, c_m\}$$

and extend the generation map of $\mathcal{A}'$ to G' to a generation map of $\mathcal{A}$ to G by sending each c_i to the equally indexed r_i. As usual we denote the image of a word w over $\mathcal{A}$ under the usual homomorphism μ by $\overline{w}$. Now define

$$L = L'c_1 \cup L'c_2 \cup \cdots \cup L'c_m.$$

Then L is regular and maps bijectively to G. So $(\mathcal{A}, L)$ is a rational structure for G. We claim that it is an automatic structure. It suffices to prove that L has the k-fellow traveller property for some k.

With this in mind, observe that

$$r_\alpha r_\beta = w_{\alpha\beta}(\overline{a_1}, \ldots, \overline{a_p}) r_{\gamma(\alpha,\beta)} \ , \quad r_\alpha a_\beta = w'_{\alpha\beta}(\overline{a_1}, \ldots, \overline{a_p}) r_{\delta(\alpha,\beta)} \ ,$$

where $w_{\alpha\beta}, w'_{\alpha\beta}$ are words in the generators $\overline{a_1}, \ldots, \overline{a_p}$, α and β range over the obvious indices and $\gamma(\alpha, \beta)$ and $\delta(\alpha, \beta)$ lie between 1 and m. Let k' be the maximum of the lengths of these words $w_{\alpha\beta}, w'_{\alpha\beta}$. Notice that if $uc_i, vc_j \in L$ and if the distance in the Cayley graph of G between $\overline{uc_i}$ and $\overline{vc_j}$ is at most 1, then $\overline{u}$ and $\overline{v}$ differ by right multiplication by one of the $w_{\alpha\beta}, w'_{\alpha\beta}$. Hence by Lemma 3 of II.B.2, remembering that $(\mathcal{A}', L')$ is an automatic structure for G', u and v are k''-fellow travellers in the Cayley graph of G', for some k''. But this implies that uc_i and vc_j are $(k'' + 2)$−fellow travellers in the Cayley graph of G. This proves one part of the the theorem.

We are left with the proof that if G is automatic then so too is G'. Let $(\mathcal{A}, L)$ be an automatic strucure for G where $\mathcal{A} = \{a_1, \ldots, a_q\}$. As before we decompose G into cosets modulo G':

$$G = G'r_1 \cup G'r_2 \cup \cdots \cup G'r_m$$

where again we take $r_1 = 1$. If $g \in G$ lies in the coset $G'r_i$, we term r_i the coset representative of g which we shall also denote by $\tilde{g}$. Put $\overline{a_j} = x_j$ and consider the elements

$$\sigma(r_i, x_j) = r_i x_j \widetilde{r_i x_j}^{-1} \ (i = 1, \ldots, m, j = 1, \ldots, q).$$

Observe that $\sigma(r_i, x_j) \in G'$. Notice also that for each choice of i and j

$$r_i x_j = \sigma(r_i, x_j) r_k$$

for a suitable choice of k. It follows from a repeated application of these mq equations that that we can express any given product $g = x_{i_1} \ldots x_{i_n}$ as a product of the $\sigma(r_i, x_j) = s_{i,j}$ and $\tilde{g}$. The process starts out by noting that $r_1 = 1$ and then proceeds as follows:

$$\begin{aligned} g = (r_1 x_{i_1}) \ldots x_{i_n} &= s_{1,i_1} \widetilde{r_1 x_{i_1}} x_{i_2} \ldots x_{i_n} \\ &= s_{1,i_1} (r_{j_1} x_{i_2}) x_{i_3} \ldots x_{i_n} \ \ (r_{j_1} = \widetilde{r_1 x_{i_1}}) \\ &= s_{1,i_1} s_{j_1,i_2} \cdots \\ &= s_{1,i_1} s_{j_1,i_2} s_{j_2,i_3} \ldots s_{j_{n-1},i_n} \tilde{g}. \end{aligned}$$

So if $g \in G'$, then $\tilde{g} = 1$ and we have re-expressed g as a product of the $s_{i,j}$. In other words G' is generated (as a monoid) by the $s_{i,j}$. This method of re-expressing g goes back to Schreier (cf. e.g. the book by Magnus, Karrass and Solitar [MKS]).

We now choose $\mathcal{A}' = \{t_{i,j} \mid i = 1, \ldots, m,\ j = 1, \ldots, q\}$ to be a set in a one-to-one correspondence with the elements of the set consisting of the $s_{i,j}$. Consider now the set of those words $w = a_{i_1} \ldots a_{i_n}$ over $\mathcal{A}$ such that $\overline{w} \in G'$. Notice that $\overline{w} = x_{i_1} \ldots x_{i_n}$ and so it can be re-expressed in the form

$$\overline{w} = s_{1,i_1} s_{j_1,i_2} s_{j_2,i_3} \ldots s_{j_{n-1},i_n}.$$

We now choose L' to consist of all the corresponding words

$$t_{1,i_1} t_{j_1,i_2} t_{j_2,i_3} \ldots t_{j_{n-1},i_n}$$

over $\mathcal{A}'$. We claim that $(\mathcal{A}', L')$ is an automatic structure for G'.

We prove first that L' is regular. Let $M = (S, Y, \mathcal{A}, \tau, s_0)$ be a finite state automaton recognizing L. We build a non-deterministic finite state automaton M' which recognizes L'. The state set of M' is taken to be

$$S' = S \times \{r_1, \ldots, r_m\}.$$

M' has a single start state $(s_0, 1)$ and the set of success states of M' is defined to be $Y \times \{r_1, \ldots, r_m\}$. We now define an action of $\mathcal{A}'$ on S' by

$$(s, r_i)t_{i,j} = (sa_j, r_k), \text{ where } r_k = \widetilde{r_i x_j}.$$

Notice that in all other cases the action of $t_{i,j}$ on S' is undefined. Then it follows that L' is the language of M' as desired.

It suffices then in order to complete the proof of the theorem to prove that L' has the $k-$fellow traveller property. But this follows without difficulty from the fact that L has the k-fellow traveller property.

5. Isoperimetric inequalities, finite presentations and the word problem.

Suppose that G is a group given by the finite presentation

$$G =< X; R >$$

where R is closed under inverses. Let F be free on X and suppose that $w \in F$ is a consequence of R, i.e. $\overline{w} = 1$ in G. Thus

$$w = p_1 r_1 p_1^{-1} \dots p_n r_n p_n^{-1} (p_i \in F, r_i \in R).$$

Then it follows from the method of disc diagrams (see, e.g. Lyndon and Schupp [LS]), which we will discuss more fully in III.7, that we can re-express w in the form

$$w = u_1 s_1 u_1^{-1} \dots u_N s_N u_N^{-1} \ (u_i \in F, s_i \in R)$$

where

$$\ell(u_i) \leq \sum_{i=1}^{N} \ell(r_i) + \ell(w).$$

Recall that a function $f : \mathbb{N} \to \mathbb{R}$ is called a Dehn function for the presentation if the u_i and s_i above can be chosen so that $N \leq f(\ell(w))$.

It is not hard to deduce the following lemma.

Lemma 1. *Let G be a group given by the finite presentation*

$$G =< X; R > .$$

Then G has a solvable word problem if and only if G has a recursive Dehn function.

It is perhaps worth noting that if one finite presentation of a group G has a recursive Dehn function, then every finite presentation of G has a recursive Dehn function.

We prove next the

Theorem 1. *Let G be an automatic group. Then*

(1) *G is finitely presented;*
(2) *G satisfies a quadratic isoperimetric inequality;*
(3) *G has a solvable word problem.*

Proof. We start out by proving that G is finitely presented. By definition G is finitely generated. So there exists a free group F freely generated by a finite set $\mathcal{X}$ together with a surjective homomorphism μ, say, of F to G. Our objective is to prove that the kernel K of μ is the normal closure of a finite set.

As in I.5, put $\mathcal{A} = \mathcal{X} \cup \mathcal{X}^{-1}$. Then, again as in I.5, $\mathcal{A}$ is a finite set of monoid generators of G. Since G is automatic, it has an automatic structure with uniqueness $(\mathcal{A}, L)$ over $\mathcal{A}$, by Theorem 1 of II.B.3. We denote the Cayley graph of G relative to $\mathcal{X}$ by Γ. As before we denote the image of $w \in F$ under μ by $\overline{w}$. Suppose that

$$w = b_1 \dots b_n \ (b_i \in \mathcal{A})$$

is a relation in G, i.e. $\overline{w} = 1$. Let

$$w_i = b_1 \dots b_i$$

be the $i-th$ initial segment of w. So $w_0 = e$ and $w_n = w$. Let u_i be the representative of w_i in L. Now view each word over $\mathcal{A}$ as an element of F in the obvious way. Then, working in F, we find that

$$(u_0 b_1 u_1^{-1})(u_1 b_2 u_2^{-1}) \dots (u_{n-1} b_n u_n^{-1}) = u_0 b_1 b_2 \dots b_n u_n^{-1} = u_0 w u_n^{-1}.$$

Since $d(\overline{u_i}, \overline{u_{i+1}}) \leq 1$, u_i, u_{i+1} are k-fellow travellers in Γ for some k. In particular there is a path p_t in Γ of length at most k from $u_i(t)$ to $u_{i+1}(t)$ for each t. Each such path p_t can be identified with a word over $\mathcal{A}$ of length at most k. Now suppose that

$$u_i = \alpha_1 \dots \alpha_\ell, u_{i+1} = \beta_1 \dots \beta_m$$

where the $\alpha_i, \beta_i \in \mathcal{A}$. For definiteness suppose that $\ell \leq m$. Then, assuming that we have chosen $p_m = b_{i+1}$ at the outset,

$$\begin{aligned}
(u_i b_{i+1} u_{i+1}^{-1})^{-1} &= u_{i+1} b_{i+1}^{-1} u_i^{-1} \\
&= (\beta_1 p_1^{-1} \alpha_1^{-1}) \alpha_1 (p_1 \beta_2 p_2^{-1} \alpha_2^{-1}) \alpha_1^{-1} \\
&. \alpha_1 \alpha_2 (p_2 \beta_3 p_3^{-1} \alpha_3^{-1}) (\alpha_1 \alpha_2)^{-1} \\
&\dots\dots \\
&. \alpha_1 \alpha_2 \dots \alpha_{\ell-1} (p_{\ell-1} \beta_\ell p_\ell^{-1} \alpha_\ell^{-1}) (\alpha_1 \alpha_2 \dots \alpha_{\ell-1})^{-1} \\
&. \alpha_1 \alpha_2 \dots \alpha_\ell (p_\ell \beta_{\ell+1} p_{\ell+1}^{-1}) (\alpha_1 \alpha_2 \dots \alpha_\ell)^{-1} \\
&\dots\dots \\
&. \alpha_1 \alpha_2 \dots \alpha_\ell (p_{m-1} \beta_m b_{i+1}^{-1}) (\alpha_1 \alpha_2 \dots \alpha_\ell)^{-1}.
\end{aligned}$$

Each of the bracketed products involving the p_j are relators in G of length at most $2k+2$ (see Figure 1, below).

Figure 1

It follows therefore that the number of such "basic" relators is finite. Moreover we have expressed every relation w as a product of conjugates of these basic relators and u_0 and u_0^{-1}, both of which are again relators since they map to 1 under μ. This completes the proof that G is finitely presented.

Next we prove that G satisfies a quadratic isoperimetric inequality.

Using the notation above, it follows from Lemma 2 of II.B.1 that there is a constant h such that

$$\ell(u_i) \leq \ell(u_{i-1}) + h \ (i = 1, \ldots, n).$$

So if $\ell(u_0) = z$, then it follows that

$$\ell(u_i) \leq ih + z.$$

Now we proved above that

$$w = u_0^{-1}(u_0 b_1 u_1^{-1})(u_1 b_2 u_2^{-1}) \ldots (u_{n-1} b_n u_n^{-1}) u_n.$$

Moreover we proved also that each of the products $u_{i-1} b_i u_i^{-1}$ can be expressed as a product of m conjugates of what we termed basic relators, where m is the maximum of $\ell(u_{i-1})$, $\ell(u_i)$. It follows therefore that $u_{i-1} b_i u_i^{-1}$ can be expressed as a product of at most $ih + z$ conjugates of the basic relators. Hence w itself can be expressed as a product of conjugates of at most

$$2 + \sum_{i=1}^{n} (ih + z) \leq cn^2$$

relators, where c is a constant. This proves the second part of the theorem.

The last part of the theorem follows now immediately from Lemma 1 of this section.

6. Negatively curved groups are automatic.

Suppose that G is a negatively curved group. Then there exists a finite set $\mathcal{X}$ of group generators of G such that every geodesic triangle in the Cayley graph Γ of G relative to the set of generators $\mathcal{X}$ is δ-thin for some δ. Again it is worth noting that because of the characterisation of negatively curved groups in terms of linear isoperimetric inequalities, given *any* finite set of generators of G there is an appropriate choice of δ such that very geodesic triangle in the Cayley graph relative to this set of generators is $\delta-$thin as well. The following lemma holds.

Lemma 1. *Any pair of geodesics γ and γ' which begin and end a distance at most 1 apart are k-fellow travellers where $k = 4\delta + 1$.*

Proof. Suppose that γ, γ' begin at the same vertex g and end 1 apart. Then γ, γ' and an edge form a geodesic triangle. At each point t in time, we claim that $d(\gamma(t), \gamma'(t)) \leq 2\delta$. There are a number of cases to consider. Suppose first that there is a geodesic path λ of length at most δ from $\gamma'(t)$ to $\gamma(s)$. If $s \leq t$ then $s \geq t - \delta$; otherwise there is a path from $\gamma(t)$ to g which is shorter than t, contradicting the fact that γ is a geodesic. Consequently there is a path from $\gamma'(t)$ to $\gamma(t)$ of length at most 2δ (see Figure 1, Case 1, below).

Figure 1

If $s > t$ then there is a geodesic from $\gamma'(t)$ of length at most δ to one of the other two sides in the geodesic triangle drawn in Figure 1, Case 2 or Figure 1, Case 3. A similar argument to the one just given shows that again $d(\gamma(t), \gamma'(t)) \leq 2\delta$ holds in Figure 1, Case 2 and a more direct argument holds in the other case (Figure 1, Case 3).

If γ, γ' begin and end a distance 1 apart then they, together with two edges, form a geodesic quadrilateral. If we add a geodesic diagonal, say λ, to this quadrilateral, we obtain two geodesic triangles (see Figure 2). It then follows from the case above that $d(\gamma(t), \lambda(t)) \leq 2\delta$ (see Figure 2). Now moving backward along λ to $\lambda(t)$ is the same as moving forward along λ to $\lambda(u)$, say. Similarly moving backward along γ' to $\gamma'(t)$ is the same as moving forward to $\gamma'(v)$. Notice that $|u - v| \leq 1$. It follows therefore that

$$d(\lambda(t), \gamma'(t)) \leq d(\lambda(u), \gamma'(u)) + 1.$$

But $d(\lambda(u), \gamma'(u)) \leq 2\delta$. Hence $d(\gamma(t)), \gamma'(t)) \leq 4\delta + 1$.

Figure 2

Next we prove the

Theorem 1. *Let G be a negatively curved group, let $\mathcal{X}$ be a finite set of group generators of G and let $\mathcal{A} = \mathcal{X} \cup \mathcal{X}^{-1}$. Furthermore let L be the set of all geodesic words over $\mathcal{A}$. Then $(\mathcal{A}, L)$ is an automatic structure for G, i.e. negatively curved groups are automatic.*

Before giving the proof of Theorem 1, we would like to point out that Gromov defines a group with a Markov property as one with a rational structure where the language is prefix closed, i.e. closed under initial segments. He then proves that in a negatively curved group the set of geodesic words has this property ([Gr] 5.2.A, 8.5).

Proof. We follow the proof given by Thurston in [T]. Let G be the negatively group given above. As usual, let

$$\mathcal{A} = \mathcal{X} \cup \mathcal{X}^{-1}.$$

Then $\mathcal{A} = \{a_1, \ldots, a_q\}$, say. Let us put $a_0 = e$. Our objective is to prove that the set L of all geodesic words over $\mathcal{A}$ is the language of an automatic structure for

G. This will be accomplished indirectly by constructing a regular language K over $\mathcal{A}$ with the k-fellow traveller property, where $k = 4\delta + 1$ and then proving that K coincides with L.

Let $B_k(1)$ be the set of elements of G of length at most k as usual. We define a finite state automaton which will accept all geodesic words and nothing else, as follows.

The set of states is the set of all subsets of $B_k(1)$, together with a fail state f.

The initial state is $\{1\}$.

Define the transition function as follows, for $x \in \mathcal{A}$: $\tau(S, x) = f$ if $\overline{x} \in S$. Otherwise, $\tau(S, x) = \{\overline{x^{-1}ga} \mid g \in S, a \in \mathcal{A} \cup \{a_0\}\} \cap B_k(1)$.

All states except f are accept states. Notice that the identity element 1 is in each live state.

Let K be the language accepted by this automaton.

We claim that no geodesic word is rejected by this machine. To see this, let $w = x_1 \ldots x_n$ be a word which is rejected. Let S_j be the state of the machine which is reached after reading the first j letters of w. Suppose that $S_m \neq f = S_{m+1}$. Then $\overline{x_{m+1}} \in S_m$. But this implies that we can find $b_1, \ldots, b_m \in \mathcal{A} \cup \{a_0\}$ so that $\overline{x_m^{-1} \ldots x_1^{-1} b_1 \ldots b_m} = \overline{x}_{m+1}$. Hence $x_1 \ldots x_{m+1}$ in not geodesic, and perforce neither is w. Thus rejection takes place only when a word can be expressed by a shorter sequence of letters.

We show by induction that this automaton accepts only geodesic words.

The empty word is accepted, as are all words of length 1 (i.e. generators) which are not trivial in the group. Now suppose that for all accepted words, the initial segments of length $< n$ are geodesic, and proceed by induction on n.

Let $w = vay \in K$ where $\ell(v) = n - 1$ and $a \in \mathcal{A}$. Suppose that va is not geodesic, then there is a geodesic word $v' \in K$ such that $\overline{va} = \overline{v'}$, and $\ell(v') < \ell(va)$. Moreover, as v is geodesic, $\ell(v') = n - 1$ or $n - 2$. Now v and v' are k fellow travellers by Lemma 1, so for each t we have $v(t)^{-1}v'(t) \in B_k(t)$. If $\ell(v') = \ell(v)$, $\overline{a}$ is in the state reached when reading the first $n - 1$ letters of v, and so the word w is not accepted. If $\ell(v') = n - 2$, then $g = v_{n-2}^{-1}v'_{n-2} \in B_k(1)$, and $g \in S_{n-2}$, the state the machine is in after reading the first $n - 2$ letters of v. Let b be the last letter of v . As the machine accepts the word va, $\overline{b^{-1}ga^{-1}} = 1$ is in following state S_{n-1}. So $\overline{b^{-1}ga_0} = \overline{a}$ also lies in the state S_{n-1}, and hence w is not accepted.

7. Asynchronously automatic groups.

Many of the results that we proved about automatic groups can be carried over also to asynchronously automatic groups. This is the object of the present section. We will content ourselves here with sketching many of the proofs, leaving a good deal more of the work to the reader than we have up till now. We have, nevertheless, contrived to arrange things in such a way that the theorems proved in III depend only on results that we will have fully proved somewhere in this paper. Again our exposition is based on that of Cannon et. al. [CEHPT].

Suppose now that G is an asynchronously automatic group. Let $(\mathcal{A}, L)$ be an asynchronously rational structure for G, where as usual

$$\mathcal{A} = \{a_1, \ldots, a_q\}.$$

We prove first an analogue of the uniqueness property of automatic groups.

Lemma 1. *Suppose G has the asynchronously automatic structure described above. Then G has another asynchronously automatic structure $(\mathcal{A}, L')$ over $\mathcal{A}$ where L' is a regular set contained in L and every element of G has a finite number of representatives in L'.*

In fact one can choose L' in such a way that every element of G has exactly one representative in L' (see [CEHPT]).

Proof. Let M be a finite state automaton such that $L(M) = L$, let $a_0 = e$, let T_0 be a two tape automaton such that $L(T_0) = L_{a_0}$ and let M_0 be the finite state automaton associated to T_0. Let k be the number of states of T_0. If some element $g \in G$ has an infinite number of representatives in L, there are words $u, v \in L$ such that $\overline{u} = \overline{v} = g$ and $\ell(v) > k^{\ell(u)}$. Now $(u, v) \in L(T_0)$. So there is a shuffle w of $(u\epsilon, v\epsilon)$ such that $w \in L(M_0)$. Therefore the path traced out by w in the state graph of M_0 from the start state to an accept state, contains a loop based at some state $s \in S_2$ which involves only states in S_2. Let z be the word consisting of the successive labels on the edges on this loop. Then z is a subword of v:

$$v = v_1 z v_2.$$

It follows that

$$(u, v_1 z^i v_2) \in L(T_0) \ (i = 0, 1, \dots).$$

So, in particular, $\overline{v_1 v_2} = \overline{v_1 z v_2}$ and hence $\overline{z} = 1$. We now build a non-deterministic finite state automaton M'' which rejects those words in L which contain such subwords z with $\overline{z} = 1$ and accepts the other words in L. The idea is to choose the states of M'' so as to be able to keep track of all sequences of states in S_2 of length at most k, enabling us to reject the unwanted subwords z. Notice that we have used ϵ for the end of tape symbol in our two-tape automaton T_0. Since we will have need to make use of $\epsilon-$transitions here as well, we will use the symbol Υ in place of ϵ in the definition of our non-deterministic automaton. We are now in a position to define M''. First we take the set of states of M'' to be

$$S_1 \cup S_2 \cup \{(s, u) \mid s \in S_2, u \in \mathcal{A}^\star, u \neq e,\ \ell(u) < k,$$
$$su_i \in S_2 \text{ all } i,\ su_i \neq su_j\ (0 \leq i < j < k)\}.$$

Here u_i denotes the $i-th$ initial segment of u. The initial state of M'' is s_0, the initial state of M_0. The set of accept states of M'' consists of the set of accept states of M_0 together with those states of M'' of the form (s, u) such that $\tau(s, u)$ is an accept state of M_0, where here τ is the transition function of M_0. The transition function τ'' of M'' is then defined as follows:

$$\begin{aligned}
\tau''(s, \Upsilon) &= \tau(s, a) \text{ for all } a \in \mathcal{A}, \\
\tau''(s, a) &= (s, a) \text{ if } \tau(s, a) \in S_2, \tau(s, a) \neq s \\
\tau''(s, a) &= \tau(s, a) \text{ if } \tau(s, a) \in S_1 \\
\tau''((s, u), a) &= \tau(s, ua) \text{ if } sua \in S_1 \\
\tau''((s, u), a) &= (s, ua) \text{ if } \tau(s, ua) \in S_2 \\
&\qquad \text{and } \tau(s, (ua)_i) \text{ are all distinct } (i < k) \\
\tau''(s, a) &= \tau(s, a) \text{ if } s, \tau(s, a) \in S_2.
\end{aligned}$$

In all other cases τ'' is undefined. Now let M' be a finite state automaton with the same language as M''. Then

$$L_{(i)} \cap (L(M') \times L(M')) \ (i = 0, \ldots, q)$$

is aysnchronously regular by Lemma 1 of II.B.7, because each $L_{(i)}$ is asynchronously regular and $L(M')$ is regular. This completes the proof on putting $L' = L(M')$.

Now suppose that T is an asynchronous finite state automaton over $\mathcal{A}$ and let M be the finite state automaton that is associated to T. We make use here of the notation above, denoting by T_i a two tape automaton such that $L(T_i) = L_{(i)}$. Let $u = u'\epsilon, v = v'\epsilon \ (u', v' \in \mathcal{A}^\star)$ and suppose that $w \in L(M)$ is such that $\Phi(w) = (u, v)$, where Φ is the usual mapping from $L(M)$ into $(\mathcal{A} \cup \epsilon)^\star \times (\mathcal{A} \cup \epsilon)^\star$. If we again denote the $t - th$ initial segment of w by w_t, then $\Phi(w_t) = (u_{\phi(t)}, v_{\psi(t)})$, where $\phi(t), \psi(t)$ are non-decreasing functions from $\mathbb{N}$ into $\mathbb{N}$. Consequently at time t, T will have read $(u'_{\phi(t)}, v'_{\psi(t)})$ as it goes through the process of accepting (u, v). We shall avail ourselves of all of this notation in the statement of the following lemma.

Lemma 2. *Let G be an asynchronously automatic group and let $\mathcal{A} = \{a_1, \ldots, a_q\}$ be a set of monoid generators of G. Put $a_0 = e$, let $(\mathcal{A}, L)$ be an asynchronously automatic structure for G and let T_i be a two-tape automaton with $L(T_i) = L_{(a_i)}$ $(i = 0, \ldots, q)$. Suppose that k is the maximum number of states in the two-tape automata $T_0, \ldots, T_q$. Then the following hold.*

(1) *If $(u', v') \in L(T_i)$, then in the Cayley graph $\Gamma_{\mathcal{A}}(G)$ we have*

$$d(\overline{u'_{\phi(t)}}, \overline{v'_{\psi(t)}}) \leq k$$

for every t.

(2) *If $(u', v') \in L(T_i)$ then there exists a word v'' such that $(u', v'') \in L(T_i)$ and T_i reads at most k letters from v'' before reading a letter from u'.*

Proof. We start out with the proof of (1). To this end let us view $\mathcal{A} \cup \{\epsilon\}$ as a monoid generating set for G by extending the usual map from $\mathcal{A}$ into G to one from $\mathcal{A} \cup \{\epsilon\}$ into G by defining $\overline{\epsilon} = 1$. Let M_i be the finite state automaton associated to T_i. Then for each t, there is a path in the state graph of M_i from $s_0 w_t$ to an accept state, of length at most $k-1$. This implies that there is a word z over $\mathcal{A} \cup \{\epsilon\}$ of length at most $k-1$ such that $w_t z$ is accepted by M_i. It follows that there exist $z_1, z_2 \in (\mathcal{A} \cup \{\epsilon\})^\star$ such that $\overline{u_{\phi(t)} z_1} = \overline{v_{\psi(t)} z_2 a_i}$ where $\ell(z_1) + \ell(z_2) = \ell(z)$. This completes the proof of (1).

In order to prove (2), suppose that T_i reads a sequence of at least k letters from v'. Then the corresponding path in the state graph of M_i contains a loop which is defined by a subword z, say, of v' and also by the same subword of w. If we omit this subword from w, the resultant word w' is again in the language of M_i. Then $\Phi(w') = (u'\epsilon, v''\epsilon)$ and so $(u', v'') \in L(T_i)$ with $\ell(v'') < \ell(v')$. It follows from this argument that there is an accepted pair as claimed.

Next we prove a result related to one proved in Cannon et. al. [CEHPT, "characterizing asynchronous"]. They prove that a rational structure with a "monotone relation" and a "departure function" is an asynchronous automatic structure. Their departure function corresponds to the conclusion of Lemma 3. The asynchronous k-fellow traveller property used here corresponds to their monotone relation.

Theorem 1. *Let G be a group, $(\mathcal{A}, L)$ a rational structure with a finite number of representatives for each element of G. Now suppose that in addition there exists a constant k such that if $u, v \in L$ and if $\overline{u} = \overline{va_i}$ for some i, then u and v are asynchronous k-fellow travellers, where as usual $\{a_1, \ldots, a_q\} = \mathcal{A}$. Then $(\mathcal{A}, L)$ is an asynchronously automatic structure for G.*

Recall that two words $u, v \in \mathcal{A}^\star$ are said to be asynchronous k-fellow travellers when there exist monotone functions ϕ and ψ (depending on u and v) such that

$$d\Big(u(\phi(t)), v(\psi(t))\Big) \leq k \ (t \geq 0),$$

where here d denotes the distance in the Cayley graph Γ of G relative to the set $\mathcal{A} \cup \{\epsilon\}$ of generators.

Theorem 1 together with Lemma 2, (1), is the counterpart for asynchronously automatic groups of the characterisation of automatic groups in terms of k-fellow travellers. It can be used to obtain the corresponding theorems for asynchronously automatic groups as those already proved for automatic groups. Thus for example, one can prove that *the direct product and the free product of two asynchronously automatic groups is again asynchronously automatic*, that *the property of being asynchronously automatic is independent of the choice of generating set* and *that a group is asynchronously automatic if and only if every subgroup of finite index is asynchronously automatic.*

We need one more fact before we can prove Theorem 1.

Lemma 3. *Suppose that $(\mathcal{A}, L)$ is a rational structure for the group G and that each element of G has only finitely many representatives in L. Given any fixed positive integer h there are at most finitely many z such that $xzy \in L$ for some choice of x and y and $d(1, \overline{z}) \leq h$.*

Proof. Suppose that the contrary conclusion holds. Then there exist infinitely many distinct z_i such that $\overline{z_1} = \overline{z_2} = \ldots$ and $x_i z_i y_i \in L$ for a suitable choice of x_i, y_i. Consequently there exist infinitely many such z_i, say $z_1, z_2, \ldots$ which begin at the same state and end at the same state in the state graph of a finite state automaton M with $L(M) = L$. It follows then that the words $x_1 z_1 y_1, x_1 z_2 y_1, \ldots$ all belong to L. But $\overline{x_1 z_1 y_1} = \overline{x_1 z_2 y_1}, \ldots$, contradicting the assumption that each element of G has finitely many representatives in L. This completes the proof of the lemma.

We are now in a position to prove Theorem 1.

Proof of Theorem 1. We must show that $L_{(=)}$ and $L_{(i)}$ are asynchronously regular for each i. Now let

$$J(i) = \{(u, v) \mid u, v \in \mathcal{A}^\star, \overline{u} = \overline{va_i}\}.$$

Then

$$L_{(i)} = J(i) \cap (L \times L).$$

It suffices therefore, by Lemma 1 of II.A.7, to prove that $J(i)$ is asynchronously regular. We will accomplish this by building a two tape automaton T_i such that $L(T_i) = J(i)$. To this end observe that if $u, v \in \mathcal{A}^\star$ are asynchronous k-fellow travellers then, by definition, there exist monotone functions ϕ and ψ (depending on u and v) such that

$$d(u(\phi(t)), v(\psi(t))) \leq k \ (t \geq 0),$$

where here d denotes the distance in the Cayley graph Γ of G relative to the set $\mathcal{A}$ of generators, as explained previously.

Now suppose that for some pair of integers m and n,

$$d(u(m), v(n)) \leq k,$$

with $m = \phi(t_0)$, $n = \psi(t_1)$. We claim that there is a bound K independant of u, v, m, n so that $|\psi(t_1) - \psi(t_0)| \leq K$. To see this, notice that $d(u(\phi(t_0)), v(\psi(t_0))) \leq k$ so that $d(v(\psi(t_0)), v(\psi(t_1))) \leq 2k$. But there are finitely many subwords of accepted words representing any element of length $2k$ by lemma 3. We take K to be the length of the longest of these subwords. We will also set K' to be the length of the longest word which represents an element of length $2k + K + 1$ in the group, and is a subword of some accepted word.

We will use this property to build our two tape automaton T_i in much the same way as we built the comparator automata before in the proof of Theorem 1 of II.B.2. T_i will be defined using the ball $B = B_{k+K+K'+1}(1)$ of radius $k+K+K'+1$ centered at 1 in Γ. Roughly speaking T_i operates as follows. Given a pair (u, v), the machine starts by reading $K+1$ letters from v; now $K+1 \geq d(u(0), v(K+1)) > k$. The machine then reads m letters from u, where m is the smallest number such that $d(u(m), v(K+1)) = k$. By lemma 3 and the above discussion, we know that $m \leq K'$.

We now show how the machine should proceed from such a situation. Suppose that while reading u, the machine reaches a point at which it has read m letters of u and n letters of v and discovers that $d(u(m), v(n)) = k$. The machine continues by reading $K+1$ letters from v. We take $n' = n + K + 1$. By the above discussion, there is $m' > m$ so that $d(u(m'), v(n')) = k$, and $m' - m \leq K'$. Thus the machine continues by reading at most K' letters of u and discovering the smallest $m' > m$ such that $d(u(m'), v(n')) = k$. If at any time an end of tape symbol is encountered (necessarily on the right hand tape), T_i switches to the other tape and continues to read it until the second end of tape symbol is finally read.

More precisely, we define the finite state automaton M_i associated to T_i as follows. The set of states S of M_i is defined as follows:

$$S = B \times \{\ell, \ell', 0, \ldots, K+1\} \cup \{f, A\}$$

where f is a fail state, and A is the unique accept state. We define then

$$S_1 = B \times \{\ell, \ell'\}, S_2 = B \times \{0, \ldots, K\} \cup \{f, A\}.$$

The transition function τ of M_i is defined as follows, where throughout $a \in \mathcal{A}, g \in B$.

$$\tau((g, \ell), a) = (\overline{a}^{-1} g, \ell) \text{ if } \ell(\overline{a}^{-1} g) > k$$
$$\tau((g, \ell), a) = (\overline{a}^{-1} g, 0) \text{ if } \ell(\overline{a}^{-1} g) = k$$
$$\tau((g, r), a) = (g\overline{a}, r+1) \text{ if } 0 \leq r < K$$
$$\tau((g, K), a) = (g\overline{a}, \ell) \leq k$$
$$\tau((g, r), a) = f \text{ if } \ell(g\overline{a}) > k + K + K' + 1$$
$$\tau((g, \ell), \epsilon) = (g, r')$$
$$\tau((g, r), \epsilon) = (g, \ell') \text{ if } 0 \leq r \leq K$$
$$\tau((g, \ell'), a) = (\overline{a}^{-1} g, \ell')$$
$$\tau((\overline{a_i}, \ell'), \epsilon) = A.$$

As always the fail state f is always taken to itself by the transition function. The start state of M_i is $(1,0)$ and A is the unique accept state. For the equality language $L_{(=)}$, the only state leading to the accept state is $(1,\ell')$. This completes the proof of Theorem 1.

We prove next the analogue for asynchronously automatic groups of Theorem 1 of II.B.5.

Theorem 2. *Let G be an asynchronously automatic group. Then the following hold:*

(1) *G is finitely presented;*
(2) *G satisfies an exponential isoperimetric inequality;*
(3) *G has a solvable word problem.*

We will closely follow the proof of the corresponding theorem, Theorem 1 of II.B.5, for automatic groups. Let us therefore assume that $\mathcal{X}$ is a finite set of group generators for G and that $\mathcal{A} = \mathcal{X} \cup \mathcal{X}^{-1}$. G has an asynchronously automatic structure $(\mathcal{A}, L)$ over $\mathcal{A}$ since the property of being asynchronously automatic is independent of the choice of generators. Let $\mathcal{A} = \{a_1, \ldots, a_q\}$, $a_0 = e$ and let T_i be a two tape automaton with $L(T_i) = L_{(a_i)}$. Let k be the maximum number of states in these two-tape automata. Now suppose that

$$w = b_1 \ldots b_n \in \mathcal{A}^\star$$

is a relator in G. We choose representatives u_i of $b_1 \ldots b_i$ for $i = 0, \ldots, n$ as follows. First we take u_0 to be a word representing the identity element of G. Next, suppose that u_i has already been chosen. Then we choose u_{i+1} so that $(u_i, u_{i+1}) \in L(T_j)$, where here j is defined by $a_j = b_i^{-1}$, and so that T_j reads at most k-letters from u_{i+1} before reading one from u_i. Notice that this choice is always possible because of Lemma 2 of this section. Since u_i and u_{i+1} are asynchronous k-fellow travellers, $\ell(u_{i+1}) \le k\ell(u_i) + 1$. It follows that for each i,

$$\ell(u_i) \le k^i(\ell(u_0) + 1).$$

We now simply follow the rest of the argument as detailed in the proof of Theorem 1 of II.B.5 to complete the proof of the other parts of the theorem.

Part III. Amalgamated products of negatively curved and automatic groups

1. Amalgamated products of automatic groups.

We recall first some facts about generalised free products. *A group G is said to be an amalgamated product of its subgroups X and Y with the subgroup Z amalgamated*, or *a generalised free product of X and Y amalgamating Z* if the following conditions hold:

(1) G is generated by $X \cup Y$;
(2) $X \cap Y = Z$;
(3) every "strictly alternating" $X \cup Y$ product

$$x_1 y_1 \ldots x_n y_n \neq 1 \ (x_i \in X - Z,\ y_i \in Y - Z).$$

It follows that if we choose a right transversal S_1 of Z in X, i.e. a complete set of representatives of the right cosets xZ of Z in X containing the element 1, and similarly a right transversal S_2 of Z in Y and if we put $S = S_1 \cup S_2$, then every element $g \in G$ can be expressed *uniquely* as a strictly alternating product

$$g = s_1 \ldots s_n z \ (s_i \in S - 1, z \in Z).$$

By this we mean that if $s_i \in S_1$ then $s_{i+1} \in S_2$ and similarly if $s_i \in S_2$ then $s_{i+1} \in S_1$. Such a form is referred to as the *normal form for g*. Of course this form depends on the choice of S_1 and S_2. We refer the reader to the book by Lyndon and Schupp [LS] for more details. We express the fact that G is an amalgamated product of X and Y with Z amalgamated by writing

$$G = X \star_Z Y.$$

It is not hard to formulate some rather general conditions which ensure that an amalgamated product of two automatic groups is again either automatic or aysnchronously automatic. The proofs of all of our other theorems about amalgamated products then comprise the verification that an appropriate subset of these conditions hold in each instance.

We begin by recalling from the introduction that a subgroup Z of a group X is an $L(X)$-rational subgroup of X for the rational structure $(\mathcal{X}, L(X))$ for X (or more simply a rational subgroup if $L(X)$ is understood), if $\mu^{-1}(Z) \cap L(X)$ is regular over $\mathcal{X}$.

Suppose now that we denote the *set* of right cosets xZ of Z in X by X/Z. We term a regular set $L(X/Z)$ contained in $L(X)$ a *regular language with uniqueness for X/Z* if the mapping

$$w \mapsto \overline{w}Z \ (w \in L(X/Z))$$

is a bijection between $L(X/Z)$ and X/Z.

The following theorem then holds.

Theorem A. *Let G be the generalised free product of the automatic groups X and Y amalgamating Z:*

$$G = X \star_Z Y.$$

Let $\mathcal{X}$ be a finite set of monoid generators for X, let $\mathcal{Y}$ be a finite set of monoid generators for Y, let $(\mathcal{X}, L(X))$ be an automatic structure for X and let $(\mathcal{Y}, L(Y))$ be an automatic structure for Y. Suppose that the following conditions hold:

(1) *Z is an $L(X)$-rational subgoup of X (and hence there is a regular language $L(Z) \subseteq L(X)$ with exactly one representative for each element of Z);*

(2) *there is a regular language $L(X/Z)$ with uniqueness for X/Z, contained in $L(X)$ and a regular language $L(Y/Z)$ with uniqueness for Y/Z contained in $L(Y)$;*
(3) *there is a constant k such that whenever $u \in L(Z)$ and $v \in L(Y)$ represent the same element of Z, then u and v are k-fellow travellers in $\Gamma_{\mathcal{X}\cup\mathcal{Y}}(G)$;*
(4) *there is a constant k such that if $u \in L(X/Z)$, if $v \in L(Z)$ and $w \in L(X)$ is such that $\overline{uv} = \overline{w}$ then uv and w are k-fellow travellers in $\Gamma_{\mathcal{X}}(X)$; and similarly if $u \in L(Y/Z)$, if $v \in L(Z)$, $w \in L(Y)$ and $\overline{uv} = \overline{w}$ then uv and w are k-fellow travellers in $\Gamma_{\mathcal{X}\cup\mathcal{Y}}(G)$.*

Then G is automatic. If (3) and (4) are replaced by

(3′) *there is a constant k such that whenever $u \in L(Z)$ and $v \in L(Y)$ represent the same element of Z, then u and v are asynchronous k-fellow travellers in $\Gamma_{\mathcal{X}\cup\mathcal{Y}}(G)$;*
(4′) *there is a constant k such that if $u \in L(X/Z)$, if $v \in L(Z)$ and $w \in L(X)$ is such that $\overline{uv} = \overline{w}$ then uv and w are asynchronous k-fellow travellers in $\Gamma_{\mathcal{X}}(X)$; and similarly if $u \in L(Y/Z)$, if $v \in L(Z)$, $w \in L(Y)$ and $\overline{uv} = \overline{w}$ then uv and w are asynchronous k-fellow travellers in $\Gamma_{\mathcal{X}\cup\mathcal{Y}}(G)$.*

Then G is asynchronously automatic.

Proof. As we noted in II.B.2 , we can assume, without any loss of generality, that all of the languages above contain the empty word e. In addition the statement that we can choose $L(Z)$ so that it contains exactly one representative for each element of Z can be justified by appealing to Proposition 1 of II.A.5 (cf. also the beginning of the proof of Proposition of II.B.1), i.e. by taking the lexicographically least elements in the regular language for Z that is guaranteed by the $L(X)-$rationality of Z.

We note that (1) and (3) or (1) and (3′) imply that Z is $L(Y)$ rational. For one may build a finite state automaton or (in the asynchronous case) a two tape automaton whose language is $\{(u,v) \mid u \in L(Y),\ v \in L(Z),\ \text{and } \overline{u} = \overline{v}\}$. This machine is based on the machine for $L(Z)$ together with a finite neighborhood of the identity in Y. We omit the details of the construction. Projection onto the first factor of this regular (or asynchronously regular) language gives a regular language, and this is none other than the set of words in $L(Y)$ with values in Z.

We now put $\mathcal{A} = \mathcal{X} \cup \mathcal{X}^{-1}$, $\mathcal{B} = \mathcal{Y} \cup \mathcal{Y}^{-1}$,

$$L^-(X/Z) = L(X/Z) - \{e\},\ L^-(Y/Z) = L(Y/Z) - \{e\}$$

and

$$R = L^-(X/Z) \cup L^-(Y/Z).$$

Then it follows from the remarks above about normal forms for the elements of G, that the set L of words of the form

$$r_1 \dots r_m u$$

satisfying the following conditions

(1) $r_i \in R\ (i = 0, \dots, m),\ u \in L(Z)$;
(2) if $r_i \in L^-(X/Z)$ then $r_{i+1} \in L^-(Y/Z)$ and if $r_i \in L^-(Y/Z)$ then $r_{i+1} \in L^-(X/Z)$

maps bijectively onto G. We refer to these words in L as strictly alternating $R-$products. Now observe that

$$L = L(X/Z)\Big(L^{-}(Y/Z)L^{-}(X/Z)\Big)^{\star} L(Y/Z)L(Z).$$

Consequently L is a regular language with uniqueness for G. Hence $L_{=} = \Delta(L)$ is regular. Our objective is to prove that, in the synchronous case, the sets L_x are regular and, in the asynchronous case, the sets $L_{(x)}$ are asynchronously regular ($x \in \mathcal{A} \cup \mathcal{B}$). The proofs are very similar. In both cases we use the $k-$fellow traveller property to prove that $(\mathcal{A} \cup \mathcal{B}, L)$ is either an automatic structure or an aysnchronously automatic structure as the case dictates. Thus in the asynchronous case we appeal to Theorem 1 and Lemma 2 (1) of II.B.7, while in the synchronous case we appeal instead to Theorem 1 of II.B.2. We will deal here with the synchronous case and leave the other to the reader. For definiteness we assume that the k specified in (3) and (4) has been chosen large enough to ensure that $L(X)$ and $L(Y)$ have respectively the $k-$fellow traveller property in the Cayley graphs $\Gamma(X)$ of X relative to $\mathcal{X}$ and $\Gamma(Y)$ of Y relative to $\mathcal{Y}$.

Thus our objective is to prove that if $c \in \mathcal{X} \cup \mathcal{Y}$ and if $w, w' \in L$ are such that $\overline{w} = \overline{w'c}$ then w and w' are $3k$-fellow travellers in Γ, where k is the constant above. The proof is divided up into a number of cases, which depend on the form of w and whether $c \in \mathcal{X}$ or $c \in \mathcal{Y}$.

For definiteness let us express w and w' as alternating $R-$products:

$$w = r_1 \dots r_m u\,,\ w' = s_1 \dots s_n v$$

where here the $r_i, s_i \in R$ and $u, v \in L(Z)$.

Case 1. $m = 0$ and $c \in \mathcal{X}$.

If $n = 0$, then $\overline{u} = \overline{vc}$ holds in X. Therefore $u = w$ and $v = w'$ are k-fellow travellers in $\Gamma_{\mathcal{X}}(X)$ and hence also in $\Gamma(G)$. If $n = 1$, then $\overline{s_1 vc} = \overline{u}$. Choose $w_1 \in L(X)$ so that $\overline{s_1 v} = \overline{w_1}$. Then by condition (4), $s_1 v$ and w_1 are $k-$fellow travellers in Γ. Now u and w_1 both lie in the language of X and $\overline{u} = \overline{w_1 c}$. So w_1 and u are $k-$fellow travellers in $\Gamma(X)$. Consequently $w' = s_1 v$ and $w = u$ are $2k-$fellow travellers in $\Gamma(G)$.

Case 2. $m = 0$ and $c \in \mathcal{Y}$.

Since $L(Z) \subseteq L(X)$ it is clear that the conditions in the statement of Theorem A are not symmetrical in X and Y. So it makes sense to treat this case in detail also. We proceed as above. If $n = 0$, then $\overline{u} = \overline{vc}$. Choose $w_1 \in L(Y)$ with $\overline{w_1} = \overline{v}$. Then v and w_1 $k-$fellow travel in Γ, by condition (3). Choose next $w_2 \in L(Y)$ so that w_2 represents the $\mathcal{Y}-$word $w_1 c$. Then w_1 and w_2 are $k-$fellow travellers in $\Gamma(Y)$. Consequently v and w_2 are $2k-$fellow travellers in Γ. But $w_2 \in L(Y)$ and $\overline{w_2} = \overline{u}$. So, by condition (3), w_2 and u are $k-$fellow travellers in Γ. But this then means that $w' = v$ and $w = u$ are $3k-$fellow travellers in Γ.

Cases 1 and 2 actually cover all the salient points in the proof of Theorem A. We will, however, detail the remaining cases and deal with them as is appropriate here.

Case 3 (a). $m > 0,\ r_m \in L(X/Z),\ s_n \in L(Y/Z),\ c \in \mathcal{X}$.

It follows that $m-1=n, r_1=s_1,\ldots,r_{m-1}=s_{m-1}$ and

$$\overline{r_m u}=\overline{vc}.$$

Now choose $w_1 \in L(X)$ so that $\overline{w_1}=\overline{r_m u}$. By condition (4) $r_m u$ and w_1 are $k-$fellow travellers in Γ. Now $\overline{w_1}=\overline{vc}$ and $v \in L(X)$. So w_1 and v are $k-$fellow travellers in $\Gamma(X)$. Hence $r_m u$ and v are $2k-$fellow travellers in Γ. But this implies also that w and w' are $2k-$fellow travellers in Γ.

Case 3 (b). $m>0,\ r_m \in L(X/Z),\ s_n \in L(X/Z),\ c \in \mathcal{X}$.

It follows here that $m=n, r_1=s_1,\ldots,r_{m-1}=s_{m-1}$ and

$$\overline{r_m u}=\overline{s_m vc}.$$

Let $w_1 \in L(X)$ be a representative of $s_m v$. Then by condition (4), $s_m v$ and w_1 are $k-$fellow travellers. Let $w_2 \in L(X)$ be a representative of $w_1 c$. Then w_1 and w_2 are $k-$fellow travellers in $\Gamma(X)$. But $\overline{w_2}=\overline{r_m u}$. So by condition 4, w_2 and $r_m u$ are $k-$fellow travellers in Γ. Therefore $s_m v$ and $r_m u$ are $3k-$fellow travellers in Γ. Hence w and w' are also $3k-$fellow travellers in Γ.

Case 4 (a). $m>0,\ r_m \in L(Y/Z),\ s_n \in L(Y/Z),\ c \in \mathcal{Y}$.

This case is very similar to Case 3 (b) above, except that we have to use condition (3) at some stage in the proof, as we did in Case 2, above. That is, for $v \in L(Z)$, we first find $v' \in L(Y)$ such that $\overline{v}=\overline{v'}$. By condition (3), v and v' are k-fellow travellers.

Case 4 (b). $m>0,\ r_m \in L(Y/Z),\ s_n \in L(X/Z),\ c \in \mathcal{Y}$.

Here the argument is analogous to that given in Case 3 (b), again maiking use of condition (3), as in Case 4 (a).

The cases considered above are the only ones that can arise and so this completes the proof of the theorem.

The only point that has to be made about the corresponding proof in the asynchronous case is that if w and w' are asynchronous $k-$fellow travellers in Γ and if w' and w'' are also asynchronous $k-$fellow travellers in Γ, then w and w'' are asynchronous $2k-$fellow travellers in Γ. The Theorem follows on application of Theorem 1 of II.B.7.

2. Amalgamated products of abelian groups.

We now use Theorem A to deduce the following

Theorem B. *Let G be the generalised free product of the finitely generated abelian groups X and Y amalgamating Z:*

$$G=X \star_Z Y.$$

Then G is automatic.

Proof. It follows from the basis theorem for finitely generated abelian groups, that Z is a direct factor of a subgroup X_1 of finite index in X and similarly a direct factor of a subgroup Y_1 of finite index in Y:

$$X_1=Z\times H,\ Y_1=Z\times K.$$

Let $(\mathcal{Z}, L(Z))$, $(\mathcal{H}, L(H))$ and $(\mathcal{K}, L(K))$ be automatic structures with uniqueness for Z, H, K respectively, chosen so as to contain e. Furthermore let $s_1 = 1, \ldots, s_m$ and $t_1 = 1, \ldots, t_n$ be complete sets of representatives of the cosets of X_1 in X and Y_1 in Y, respectively. Let

$$\mathcal{A} = \{a_2, \ldots, a_m\} \cup \mathcal{H} \cup \mathcal{Z}$$

and

$$\mathcal{B} = \{b_2, \ldots, b_n\} \cup \mathcal{K} \cup \mathcal{Z}.$$

$\mathcal{A}$ and $\mathcal{B}$ can be viewed as monoid generating sets for X and Y respectively in the obvious way, with the a_i mapping to the s_i and the b_i mapping to the t_i. We now put

$$L(X) = \bigcup_{i=1}^{m} a_i L(H) L(Z), \;\; L(Y) = \bigcup_{i=1}^{n} b_i L(K) L(Z)$$

where we define $a_1 = b_1 = e$. Then $L(X)$, $L(Y)$ are languages with uniqueness for X and Y respectively. It follows immediately that Z is an $L(X)-$rational subgroup of X and that we can take $L(X/Z) = \bigcup_{i=1}^{m} a_i L(H)$ and $L(Y/Z) = \bigcup_{i=1}^{m} b_i L(K)$. The conditions laid down in Theorem A are immediately satisfied and therefore the proof of Theorem B is complete.

3. Quasiconvexity and negatively curved groups.

Before turning our attention to amalgamated products of negatively curved groups we need to introduce some additional notions. To this end let G be a group and let $\mathcal{A}$ be a monoid set of generators of G. We can therefore think of $\mathcal{A}$ as a set of group generators of G. Then $\Gamma_{\mathcal{A}}(G)$ is, as usual, the Cayley graph of G with respect to the given set $\mathcal{A}$ of group generators of G. A geodesic γ in Γ can be represented by a word w over $\mathcal{A} \cup \mathcal{A}^{-1}$. We sometimes refer to such a word as a *geodesic word*. We define $\gamma(t) = w(t)$ for every $t \geq 0$ and $\overline{\gamma} = \overline{w}$. If $g \in G$, then there exists a shortest word w over $\mathcal{A} \cup \mathcal{A}^{-1}$ with $\overline{w} = g$. We define the *length* $|g|_{\mathcal{A}}$ *of* g, relative to $\mathcal{A}$, to be the length of such a word w. So $|g|_{\mathcal{A}}$ is the length of a geodesic word over $\mathcal{A} \cup \mathcal{A}^{-1}$ representing g. We denote the distance function in $\Gamma_{\mathcal{A}}(G)$ by $d_{\mathcal{A}}$ or simply by d if there is no risk of confusion. We have already discussed the notion of a rational subgroup of an automatic group. We define now, using the notation introduced above, two related concepts which turn out to coincide in the case of a negatively curved group.

Definition. *A subgroup C of G is termed quasiconvex (with respect to $\mathcal{A}$) if there is an ϵ such that*

$$d(\gamma(t), C) \leq \epsilon$$

for every geodesic γ in Γ, with $\overline{\gamma} \in C$.

Now suppose that $\mathcal{C}$ is a finite monoid set of generators of C. Then, as above, if $c \in C$, then $|c|_{\mathcal{C}}$ denotes the length of a geodesic word w over $\mathcal{C}$ with $\overline{w} = c$. This allows us to formulate our second definition.

Definition. *Let G and C be the groups described above. We term C quasigeodesic (with respect to $\mathcal{A}$ and $\mathcal{C}$) if there exists a positive real number λ such that*

$$|g|_{\mathcal{A}} \geq |g|_{\mathcal{C}}/\lambda$$

for every $g \in C$.

The notion of quasiconvexity may well be dependent on the choice of generating set in some instances. However in the case of negatively curved groups this is not the case, because of Theorem 1, below. On the other hand, being quasigeodesic is always independent of the choice of the generating sets $\mathcal{A}$ and $\mathcal{C}$. For if $\mathcal{A}'$ and $\mathcal{C}'$ are a second pair of monoid generating sets for G and C respectively, then there are constants k_1 and k_2 such that

$$|g|_{\mathcal{A}'}(g) \geq |g|_{\mathcal{A}}/k_1\,,\ k_2|g|_{\mathcal{C}} \geq |g|_{\mathcal{C}'}(g).$$

It follows that if C is quasigeodesic with respect to $\mathcal{A}$ and $\mathcal{C}$ then it is also quasigeodesic with respect to $\mathcal{A}'$ and $\mathcal{C}'$ with constant $\lambda' = k_1k_2\lambda$.

The following theorem then holds.

Theorem 1. *Let G be a negatively curved group and let $\mathcal{A}$ be a finite set of monoid generators of G. Furthermore, let C be a subgroup of G and suppose that $\mathcal{C}$ is a set of monoid generators of C which is contained in $\mathcal{A}$. Let L be the set of all geodesics in the Cayley graph $\Gamma = \Gamma_{\mathcal{A}}(G)$ of G. Then the following conditions are equivalent.*

(1) *C is quasiconvex with respect to $\mathcal{A}$;*
(2) *C is quasigeodesic with respect to $\mathcal{A}$ and $\mathcal{C}$;*
(3) *C is L-rational.*

The proof of Theorem 1 depends on some results of Gromov [Gr] and Gersten and Short [GS3]. We will also need the following notion of Gromov [Gr].

Definition. *Let λ and ϵ be non-negative real numbers. Then a word (or path) $w = b_1 \dots b_n$ ($b_i \in \mathcal{A} \cup \mathcal{A}^{-1}$) in the Cayley graph $\Gamma_{\mathcal{A}}(G)$ is termed a (λ, ϵ)-geodesic if*

$$|\overline{u}|_{\mathcal{A}} \leq \ell(u) \leq \lambda|\overline{u}|_{\mathcal{A}} + \epsilon$$

for every subword $u = b_i \dots b_j$ $(0 \leq i \leq j \leq n)$ of w. If $\epsilon = 0$ we refer to w as a λ-quasigeodesic.

The following result of Gromov [Gr] holds (cf [GH], [CDP], [ABC]).

Lemma 1. *Let $\mathcal{A}$ be a finite set of monoid generators of the negatively curved group G and let Γ denote the corresponding Cayley graph of G. Then for each pair of non-negative real numbers λ, ϵ there exists a non-negative real number $\epsilon'(\lambda, \epsilon, \delta)$ such that every (λ, ϵ)-quasigeodesic w lies in an ϵ'-Hausdorff neighbourhood of every geodesic v in Γ for which $\overline{v} = \overline{w}$ and conversely.*

Proof. We give an outline of the proof given in [ABC]. The proof relies on the following Proposition which is in fact another characterization of negatively curved spaces.

Proposition. *Let Γ be the Cayley graph of a negatively curved group and suppose triangles in Γ are δ thin. Then there are constants $d \geq 0$ and $k > 1$ depending only on δ with the following property: Suppose γ and γ' are geodesics emanating from a common point $g \in \Gamma$, each of length $s+t$. Suppose also that $d(\gamma(t), \gamma'(t)) \geq d$. Let p be a path from $\gamma(s+t)$ to $\gamma'(s+t)$ such that $d(g,p) = s+t$. Then $\ell(p) > k^s$.*

We take $d = 2\delta + 1$. Now let α be a geodesic running from $\gamma(t+s)$ to $\gamma'(t+s)$. Let P denote the midpoint of p and let α_0 be a geodesic running from the begining of α to P. Let α_1 be a geodesic running from P to the end of α. Let b be a finite sequence of ones and zeroes, and suppose that the geodesic α_b has been defined and has its origin and terminus on p. Let P_b be the midpoint of the segment of p whose origin and terminus are those of α_b. Now let α_{b0} be a geodesic running from the origin of α_b to P_b, and let α_{b1} be a geodesic segment running from P_b to the terminus of α_b.

At each stage, we have subdivided p, and after at most $n = \log_2(\ell(P)) + 1$ such subdivisions, we have $\ell(\alpha_b) \leq 1$.

Now γ, γ' and α form a geodesic triangle. Hence $\gamma(t)$ is within δ of either γ' or α. But $d(\gamma(t), \gamma'(t)) \geq 2\delta + 1$ and consequently $d(\gamma(t), \gamma') > \delta$. Hence, there is a point v_0 on α such that $d(\gamma(t), v_0) \leq \delta$. Now for each binary sequence b (possibly empty), α_b, α_{b0} and α_{b1} form a geodesic triangle. Thus, if we have found v_m on α_b, we can find v_{m+1} on either α_{b0} or α_{b1} with $d(v_m, v_{m+1}) \leq \delta$.

Now $d(v_n, p) \leq 1$, and p lies outside the interior of the ball or radius $s+t$ around g. Hence $d(\gamma(t), v_n) \geq s - 1$. On the other hand, as the vertices $\{v_i\}$ verify, $d(\gamma(t), v_n) \leq (n+1)\delta$. But this shows that $\ell(p) \geq 2^{\frac{s}{\delta}-3}$. Hence $\ell(p)$ grows exponentially in s and k may be chosen appropriately.

Returning to our lemma, we first show that v stays close to w. To see this, let g be a point of v which acheives the maximum distance from w. Say, $d(g,w) = D$. In particular, the interior of the ball of radius D around g misses w. Let a and b be the points of v at distance $2D$ from g. (They may be taken to be the origin or terminus of v respectively if either of these points is within $2D$ of g.) We now take γ and γ' to be the segments of v running respectively from g to a and g to b. We take $p = yuz$ where u is a subpath of w, and y and z run respectively from a to w and from w to b. We may take each of y and z to have length at most D. Now if $D \geq \frac{d}{2}$ then $\ell(p)$ is exponential in D, which forces $\ell(u)$ to be exponential in D. On the other hand, $|\overline{u}| \leq 6D$, and since w and hence u is quasigeodesic, this means $\ell(v) \leq \frac{6D+\epsilon}{\lambda}$. Hence, D is bounded, since an exponential function must eventually surpass a linear one.

Let D' be the bound we have just found. Let $N = N_{D'}(v) = \{q \mid d(q,v) \leq D'\}$. Suppose w does not lie in the interior of N. Let u be a component of the closure of $w \setminus \mathrm{Int}(N)$. Then the origin and terminus of u are within D' of points a and b on v. Let x and y be the components of $w \setminus u$. Then each point of v between a and b is within D' of either x or y. But by continuity of distance and connectedness of the interval, we can find a point c between a and b so that $d(c,x) \leq D'$ and $d(c,y) \leq D'$ Let the points of x and y realizing these distances be a' and b'. Then $d(a',b') \leq 2D'$ and consequently $\ell(u) \leq \frac{2D'+\epsilon}{\lambda}$. Now each point of u is within $\frac{\ell(u)}{2} + D'$ of v, so we can take $\epsilon' = \frac{2D'+\epsilon}{2\lambda} + D'$

We are now in a position to prove Theorem 1.

First we observe that it follows from the theorem of Gersten and Short [GS3] alluded to above, that the conditions (1) and (3) are equivalent. It suffices therefore to prove that the conditions (1) and (2) are equivalent.

We prove first that (1) implies (2). We need to prove that there exists a non-negative real number λ such that for every $g \in C$, $|g|_{\mathcal{A}} \geq |g|_{\mathcal{C}}/\lambda$. By hypothesis, there exists an ϵ such that every geodesic in Γ that ends up in C stays within ϵ of C. Define

$$S = \{c \mid c \in C,\ |c|_{\mathcal{A}} \leq 2\epsilon + 1\}.$$

Then S is a finite set. We define

$$\lambda = \max\{|c|_{\mathcal{C}} \mid c \in S\}.$$

Now let $g \in C$ and let γ be a geodesic in $\Gamma_{\mathcal{A}}(G)$ of length n, say, which starts at 1 and ends at g, i.e. $\overline{\gamma} = g$. We "shadow" γ by points $p(t) \in C, 0 \leq t \leq n$ where the $p(t)$ are chosen so that $d_{\mathcal{A}}(\gamma(t), p(t)) \leq \epsilon$. In particular, we choose $p(0) = 1$ and $p(n) = g$. Observe that for $0 \leq t < n$, $d_{\mathcal{A}}(p(t), p(t+1)) \leq 2\epsilon + 1$. Therefore $d_{\mathcal{C}}(p(t), p(t+1)) \leq \lambda$. Consequently $|g|_{\mathcal{C}} \leq n\lambda$ which translates into

$$|g|_{\mathcal{A}} \geq |g|_{\mathcal{C}}/\lambda,$$

as required.

We now prove that (2) implies (1). Suppose then that γ is a geodesic in $\Gamma_{\mathcal{A}}(G)$ such that $\overline{\gamma} \in C$. Let γ' be a geodesic in $\Gamma_{\mathcal{C}}(C)$ such that $\overline{\gamma'} = \overline{\gamma}$. As each subword of γ' is a geodesic in $\Gamma_{\mathcal{C}}(C)$, the fact that the subgroup is λ-quasigeodesic means that γ' is a $(\lambda, 0)$ quasigeodesic in $\Gamma_{\mathcal{A}}(G)$. So, by Lemma 1, there exists a non-negative real number ϵ such that

$$d_{\mathcal{A}}(\gamma'(t), \gamma) \leq \epsilon \ (t \geq 0).$$

Since γ' lies in $\Gamma_{\mathcal{C}}(C)$, this completes the proof.

4. Amalgamated products of negatively curved groups.

We shall prove here our main theorem.

Theorem C. *Let X and Y be negatively curved groups and let $(\mathcal{X}, L(X))$, $(\mathcal{Y}, L(Y))$ be automatic structures for each consisting of geodesic words. Let*

$$G = X \star_Z Y$$

be an amalgamated product of X and Y amalgamating a subgroup Z that is rational in both X and Y (with respect to the languages $L(X), L(Y)$). Then G is asynchronously automatic. If, in addition, there is a constant k' such that for every $g \in Z$, and for every $w \in L(X)$, $w' \in L(Y)$ such that $\overline{w} = \overline{w'} = g$

$$|w(t) - w'(t)|_{\mathcal{X} \cup \mathcal{Y}} \leq k',$$

then G is automatic.

The proof of Theorem C depends on a number of lemmas. Before we start out on the proof of these lemmas we recall that if G is a negatively curved group, then the language consisting of all geodesic words over $\mathcal{A} \cup \mathcal{A}^{-1}$ where $\mathcal{A}$ is a monoid generating set for G, is the language of an automatic structure for G.

Choose finite sets of monoid generators $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ for X, Y, Z respectively so that

$$\mathcal{Z} \subseteq \mathcal{X}.$$

If $x \in X$ let $[x]$ denote the left coset xZ of Z in X containing x. So $[x] = [x']$ if and only if $x = x'z$ for some $z \in Z$.

We remind the reader that if $x \in X$, then the $\mathcal{X}$-length $|x|_{\mathcal{X}}$ of x is the minimum of the lengths of the words in $(\mathcal{X} \cup \mathcal{X}^{-1})^{\star}$ representing x

The following lemma then holds.

Lemma 1. *There exists a number K such that if $x, x' \in X$ and $[x] = [x']$ and each of x, x' is of minimal $\mathcal{X}-$length in $[x]$, then $x = x'z$ with $|z|_{\mathcal{Z}} \leq K$.*

Proof. Suppose that Lemma 1 is false. Then we can find x, x' of minimal length in $[x]$ such that $x = x'z$ where $z \in Z$ and $|z|_{\mathcal{Z}}$ is arbitrarily large. Since Z is a rational subgroup of X, it is quasiconvex and also quasigeodesic, by Theorem 1 of III.3. In particular there is a real number λ such that $|z|_{\mathcal{X}} \geq |z|_{\mathcal{Z}}/\lambda$. The reader should for the rest of the proof refer to Figure 1, below, as needed. Let u be a geodesic in $\Gamma_{\mathcal{X}}(X)$ running from x' to x with $\overline{u} = z$. Then u is also very long. Let c be either of the two vertices in the middle of u, if $|u|_{\mathcal{X}}$ is odd, or the midpoint of u otherwise. Since u is a geodesic, c is far from both x and x'. Let v, w be geodesics in $\Gamma_{\mathcal{X}}(X)$ running from 1 to x and x' respectively. Then v, w and u form a geodesic triangle, which is δ-thin. So c is close to at least one of v, w. Consequently there is a short (i.e. length at most δ) path p^{-1}, from c to, say, v. Let p^{-1} meet v at $v(n)$. Notice that $v(n)$ is far from x, for otherwise we would have a short path from c to x, contradicting the assumption that u is long. Since Z is quasiconvex, we can find a short path r of length less than a uniform bound ϵ from c to a geodesic h representing z in $\Gamma_{\mathcal{Z}}(Z)$. Hence if $|z|_{\mathcal{Z}}$ is sufficiently large, the path $v_n pr$ is shorter than the length of x in $\Gamma_{\mathcal{X}}(X)$, where as usual v_n denotes the $n-th$ initial segment of v. But $[v_n pr] = [x]$, which contradicts the assumption that x is of minimal length in $[x]$. This completes the proof of the lemma.

Figure 1

Next we prove

Lemma 2. *There exists a real number K' such that if x is not of minimal $\mathcal{X}-$length in $[x]$, then for some $z \in Z$ with $|z|_{\mathcal{Z}} \leq K'$, $|xz|_{\mathcal{X}}$ is less than $|x|_{\mathcal{X}}$.*

Proof. Suppose the lemma is false. Then we can find an element $x \in X$ such that all shorter elements in $[x]$ are arbitrarily far from x. In particular any shortest element x' in $[x]$ is arbitrarily far from x. The reader should refer to Figure 2 and Figure 3, below, as the proof proceeds. Let v and w be geodesic words over $\mathcal{X}$ with $\overline{v} = x, \overline{w} = x'$ and let u be a geodesic in $\Gamma_{\mathcal{X}}(X)$ from x to x'.

Notice that $\overline{u} = z \in Z$ and hence $|z|_{\mathcal{Z}}$ and consequently also $|z|_{\mathcal{X}}$ are arbitrarily large. Since there are infinitely many choices for x, it follows that we can assume also that $|x|_{\mathcal{X}}$ is arbitrarily large. This implies that we can choose n so that $v = v_n t$ where $|t|_{\mathcal{X}} = \delta + \epsilon + 1$, where δ is the constant that comes from the fact that G is negatively curved and the ϵ is the constant that arises from Lemma 1 of III.3 on taking advantage of the fact that Z is quasigeodesic with associated constant λ. There is a path p of length at most δ from $v(n)$ to either u or w. There are therefore two possibilities to consider.

Suppose first that p runs from $v(n)$ to a point c in u (see Figure 2). Let h be a geodesic in $\Gamma_{\mathcal{Z}}(Z)$ running from x to x'. Then there is a path r from c to h of length at most ϵ. Now $\overline{v_n pr} \in [x]$ and $\ell_{\mathcal{X}}(v_n pr) < |x|_{\mathcal{X}}$. But $x = \overline{v_n pr} z'$, where $z' \in Z$ and $|z'|_{\mathcal{Z}} \leq \lambda(2\delta + 2\epsilon + 1)$. This contradicts our initial assumption about x and the elements in $[x]$ which are shorter than x.

Suppose next that p runs to w, say p meets w at $w(m)$. Then we can write $w = w_m s$ (see Figure 3). So $\overline{v_n p} = \overline{w_m}$. It follows that s is very long since u is very long and p and t are short. Since w is no longer than v, it follows that $w_m p^{-1}$ is shorter than v_n, which contradicts the assumption that v_n is a geodesic. This completes the proof of the lemma.

Figure 2

Figure 3

Next we prove

Lemma 3. *Let X be negatively curved, Z a quasiconvex subgroup of X. Let $\mathcal{X}$ be a set of monoid generators for X and let L be the set of all geodesic words over $\mathcal{A} = \mathcal{X} \cup \mathcal{X}^{-1}$. Let $L(X/Z) = \{w \in L \mid w$ is lexicographically least in $[\overline{w}]\}$. Then $L(X/Z)$ is regular.*

Proof. Since $(\mathcal{A}, L)$ is an automatic structure for X, L_x is regular for every $x \in X$. Hence, making use here of the constants K and K' obtained in Lemma 1 and Lemma 2 above, so too is

$$L' = \bigcup_{|z|_Z \leq K+K'} L_z.$$

Then by Proposition 1 of II.A.5

$$\{w \mid \ell(w) \leq \ell(w') \text{ and } w \text{ is lexicographically earlier than } w' \text{ whenever } \nu(w,w') \in L'\} \blacksquare$$

is regular. But this set is simply $L(X/Z)$.

We are now in a position to prove Theorem C. Our objective is to show that Theorem A can be applied under the assumptions given in the statement of Theorem C. In view of Lemma 3 above, we are left with verifying that the conditions (3′) and (4′) of Theorem A are satisfied. We deal first with (3′).

We will need the following lemma which follows Cannon's notion of "progression in geodesic corridors." [**C**]

Lemma 4. *Let X be a negatively curved group, let $\mathcal{X}$ be a monoid set of generators of G and let $\Gamma = \Gamma_{\mathcal{X}}$ be the Cayley graph of G relative to the set $\mathcal{X}$ of generators. Let λ, ϵ be positive real numbers, let u be a (λ, ϵ) quasigeodesic word over $\mathcal{A} = \mathcal{X} \cup \mathcal{X}^{-1}$ and let v be a geodesic word over $\mathcal{A}$ such that $\overline{u} = \overline{v}$. Let δ be chosen so that geodesic triangles in Γ are δ-thin. Then there is a positive number $K = K(\lambda, \epsilon, \delta)$ depending on λ, ϵ, and δ but not on u and v such that u and v are asynchronous K-fellow travellers in Γ.*

Proof. By Lemma 1 of the previous section, there is a real number $\epsilon' = \epsilon'(\lambda, \epsilon, \delta)$ such that the (λ, ϵ)-quasigeodesic u lies in a ϵ'-neighbourhood of the geodesic v. Thus for each t, there is a value $s(t)$ such that $d(u(t), v(s(t)) \leq \epsilon'$. Now $d\big(v(s(t)), v(s(t+1))\big) \leq 2\epsilon' + 1$ as v is geodesic, and we can choose $s(0) = 0$, and $s(\ell(u)) = \ell(v)$. Notice that this means that in any closed interval $I = [\alpha, \alpha + \beta]$ of length $\beta \geq 2\epsilon'$, $(\alpha > 0)$, if $s(t) < \alpha$, there is some value of $t'' > t$ such that $s(t'') \in I$.

We now show that there is a constant $K' = K'(\lambda, \epsilon, \delta)$ such that when $t' > t + K'$, it follows that $s(t') > s(t)$. Suppose that we have $N > 0$, $t' > t+N$ and $s(t') < s(t)$. Now choose $t'' > t'$ such that $s(t) < s(t'') < s(t) + 2\epsilon' + 1$; then $d(u(t), u(t'')) < 4\epsilon' + 1$. As u is a (λ, ϵ)-quasigeodesic, this means that $|t - t''| < 4\lambda(\epsilon' + 1) + \epsilon = K'$, thus bounding N.

Let $p(t) = \max\{s(r) \mid r \leq t\}$. This is a monotone function function and for all t, $d(u(t), v(p(t)) < K' + \epsilon'$, and $p(t+1) - p(t) < 2\epsilon' + 1$. Now define the monotone functions ϕ and ψ so that $\phi(t)$ is constant during the time when ψ increases from $p(t)$ to $p(t+1)$, and ψ stays constant while ϕ increases by 1.

By construction we have that $d(\phi(t), \psi(t)) \leq K' + 3\epsilon' + 1$. This implies that u and v are asynchronous K-fellow travellers, as required.

We are now in a position to verify that condition (3′) of Theorem A is satisfied. We choose a finite set $\mathcal{Z}$ of monoid generators for Z. Now by Theorem 1 of III.3, Z is λ-quasigeodesic in X and also in Y, for a suitable choice of λ. This implies that if z is a word over $\mathcal{Z}$ which is geodesic in the Cayley graph of Z, then z is a λ-quasigeodesic in both X and also in Y. Suppose then that $u \in L(Z)$ and $v \in L(Y)$ represent the same element of Z. Choose the geodesic z above so that $\overline{u} = \overline{z} = \overline{v}$. Then by Lemma 4, there exists a K such that u and z are asynchronous K-fellow travellers in X and z and v are asynchronous K-fellow travellers in Y. Hence u and v are asynchronous $2K$-fellow travellers in G. So we can choose the k in condition (3′) to be $2K$. This then verifies that condition (3′) is satisfied.

Notice that the additional assumption of the present theorem is none other than condition (3) of Theorem A.

Before completing the proof of the theorem, we need to recall the definition of the tripod map. A geodesic triangle Δ in the Cayley graph $\Gamma_{\mathcal{X}}$, of the negatively curved group X with finite set of generators $\mathcal{X}$, is δ-thin, i.e each point on each side of Δ lies within δ of the union of the other two sides. Let the vertices of Δ be x_1, x_2, x_3, and let the sides be the geodesic segments $[x_i, x_j]$. We can find three points x'_1, x'_2, x'_3 in $\mathbb{E}^2$, the Euclidean plane with its usual metric, so that $d_\Gamma(x_i, x_j) = d_{\mathbb{E}^2}(x'_i, x'_j)$ for all $1 \le i, j \le 3$. Thus x'_1, x'_2, x'_3 are the vertices of the Euclidean triangle Δ'. Let the inscribed circle of Δ' meet the side $[x'_i, x'_j]$ at y_k, where $\{i, j, k\} = \{1, 2, 3\}$. The tripod map T_Δ associated to Δ is the identification space obtained by identifying the segments $[x'_i, y_j]$ and $[x'_i, y_j]$ by the isometry fixing x'_i, for all i, j, k. Thus T_Δ is a tree with one branch point b, the image of the points y_i, and three segments emerging from b. The composite mapping $\Delta \to \Delta' \to T_\Delta$ is called the tripod map f_Δ; here the first map $\Delta \to \Delta'$ is the evident isometry and the second map $\Delta' \to T_\Delta$ is the identification map. A basic result is that under our hypotheses on $\Gamma_{\mathcal{X}}$, a geodesic triangle Δ is 4δ-thin, that is, the fibres of the map f_Δ are of the diameter at most 4δ ([GH, page 40]). It follows that there is a constant K such that all geodesic triangles in the Cayley graph are K-fine if and only if the group is negatively curved.

We are left with the verification of condition (4′). We shall prove somewhat more, namely that condition (4) itself is satisfied. This is the content of the following lemma.

Lemma 5. *There is a constant $k > 0$ such that if $u \in L(X/Z)$, $v \in L(Z), w \in L(X)$ and $\overline{uv} = \overline{w}$ then uv and w are (synchronous) k-fellow travellers in X.*

Proof. We may assume that all geodesic triangles in Γ are 4δ-fine. This means that for each such triangle Δ the tripod map f_Δ has all fibres of diameter at most δ. Consider the geodesic triangle Δ in Γ with sides $u, \bar{u} \cdot v$, and w. Here $g \cdot v$ denotes the left translate of the path v by the group element g from the left action on Γ. The side $\bar{u} \cdot v$ begins at $\bar{u}$ and ends at $\bar{w}$. Let the points $u(t_0), \bar{u} \cdot v(\epsilon)$ and $w(t_0)$ be the inner points of Δ. Hence $\ell(w) = \ell(u) + \ell(v) + 2\epsilon$.
Recall that the inner points are the preimage of the unique branch point of the tripod graph under the tripod map f_Δ. Here $\epsilon = \ell(u) - t_0$. The diameter of the set of inner points is at most δ by the definition of δ-fineness. As Z is quasiconvex, there is a constant S such that $v(\epsilon)$ is at distance at most S from some point in Z. We note that $\epsilon \le 4\delta + S$, since the opposite inequality would contradict the choice of u as an element of least length in $L(Z)$ representing its coset $\bar{u}Z$. Note also that $\ell(w) = \ell(u) + \ell(v) - 2\epsilon = t_0 + \epsilon + \ell(v)$. We shall show that we can take K in the

statement of Lemma 5 to be $12\delta + 2S$.

First note that for $t \leq t_0$ we have $d(w(t), u(t)) \leq 4\delta$. In this interval uv agrees with u. Next note that for $t_0 \leq t \leq t_0 + \epsilon$ we have $d(u(t), w(t)) \leq 2\epsilon + 4\delta \leq 12\delta + 2S \leq K$. This follows from the fact that $u(t_0)$ and $w(t_0)$ are inner points of Δ and from the triangle inequality (since u and w are geodesics). Thirdly note that for $\ell(u) \leq t \leq \ell(u) + \epsilon = t_0 + 2\epsilon$ we have $uv(t) = \bar{u} \cdot v(t - \ell(u))$. It follows that

$$d(uv(t), w(t)) = d(\bar{u} \cdot v(t - \ell(u)), w(t)) \leq 2\epsilon + 4\delta \leq K$$

as in the case immediately preceding.

Fourthly assume that $t \in [t_0 + 2\epsilon = \ell(u) + \epsilon, \ell(uv)]$. Here we have $uv(t) = \bar{u} \cdot v(t - \ell(u))$ and consequently $d(uv(t), w(t)) = d(\bar{u} \cdot v(t - \ell(u)), w(t))$.

We know that $d(w(\ell(w) - s), \bar{u} \cdot v(\ell(v) - s)) \leq 4\delta$ for

$$0 \leq s \leq \ell(w) - t_0 = \ell(v) - \epsilon$$

by the definition of inner points. Let $s = \ell(w) - t$. But $\ell(v) - s = \ell(w) - \ell(u) - 2\epsilon - s = t - \ell(u) - 2\epsilon$, and thus we have that

$$d(uv(t), w(t)) = d(\bar{u} \cdot v(\ell(v) - s + 2\epsilon), w(\ell(w) - s)) \leq 2\epsilon + 4\delta \leq 12\delta + 2S.$$

These four cases exhaust the possibilities for t in the domain of uv, and so we have shown that uv and w are fellow travellers, as required. That is to say, we have shown that condition (4) of Theorem A holds in X.

To complete the proof of Theorem C, it remains to show that the second part of condition (4′) of Theorem is satisfied. Suppose we are given $w \in L(Y)$, $u \in L(Y/Z)$ and $v \in L(Z)$, so that $\overline{w} = \overline{uv}$. Then by condition (3′), which we established thanks to Lemma 4, there is a word $v' \in L(Y)$ which is asynchronously fellow travels v in $\Gamma_{\mathcal{X} \cup \mathcal{Y}}(G)$, such that $\bar{v} = \bar{v}'$. By the same argument as above w and uv' are fellow travellers in $\Gamma_{\mathcal{Y}}(Y)$ and hence in $\Gamma_{\mathcal{X} \cup \mathcal{Y}}(G)$. Hence w and uv asynchronously fellow travel in $\Gamma_{\mathcal{X} \cup \mathcal{Y}}(G)$. Hence condition (4′) of Theorem A is verified.

Under the stronger assumption that condition (3) holds, v and v' are synchronous fellow travellers in $\Gamma_{\mathcal{X} \cup \mathcal{Y}}(G)$ and thus, so are w and uv, so that we have verified condition (4) of Theorem A.

This completes the proof of Theorem C.

5. Cyclic amalgamations. Here we use Theorem C as the starting point in the proof that

Theorem D. *An amalgamated product of two negatively curved groups with a cyclic subgroup amalgamated is automatic.*

As we have already noted, finitely generated free groups are also negatively curved and so we have the following corollary to Theorem D.

Corollary 1. *Let G be an amalgamated product of two finitely generated free groups X and Y amalgamating a cyclic subgroup Z. Then G is automatic.*

Before we give the proof of Theorem D, we will give an independent proof of Corollary 1, which depends on related work of Gersten and Short [GS1], and which is of interest in its own right.

Proof of Corollary 1. We choose free bases for X and Y and denote the images of a generator of Z in X and Y respectively by α and β respectively. We can assume, without any loss of generality, that both α and β are given by cyclically reduced words in terms of the bases of X and Y. If either α or β is of length 1, then G is free and there is nothing to prove. Thus we suppose that this is not the case. Let J and K be 1-dimensional complexes with a single vertex and one edge for each generator of X and Y, respectively. The words α and β can be represented by immersed loops (i.e. loops without backtracking) in J and K, again denoted α and β.

We consder first the case when α and β are of equal length $k \geq 2$. Subdivide the circle S^1 into k segments by inserting k vertices. This induces a subdivision of the annulus $S^1 \times I$ into k rectangles. We then identify $S^1 \times \{0\}$ with the loop α in J via a simplicial map. Similarly we idntify $S^1 \times \{1\}$ with the loop β in K. We denote the resultant complex by K'. Then, by the well known theorem of Seifert and van Kampen, the fundamental group of K' is our group G. We now identify the two vertices of K'. The resultant complex K'' has fundamental group $G\star < t >$, the free product of G with an infinite cyclic group $< t >$ on t. Following Gersten and Short [GS1] we say that a presentation satisfies the $C''(4)$-condition if a piece common to two relators has length at most 1 and every relator has length at least 4. It follows from the construction of the complex K'' that its fundamental group can be defined by a set of relations corresponding to the rectangles in $S^1 \times I$. These relations are all of length 4 and the resultant presentation satisfies not only the $C''(4)$-condition but also the usual $T(4)$ condition of small cancellation theory. The main theorem of Gersten and Short [GS1] states that such groups are automatic. Hence it follows that G too is automatic, by Theorem E of III.6 below, since it is a free factor of an automatic group.

Suppose next that α is of length p, β is of length q and that $p \neq q$. We then subdivide every edge in J by inserting q new vertices and identify these new vertices with the single vertex of J. Similarly we subdivide every edge in K inserting p new vertices and then again identifying them with the single vertex of K. In each instance the new complexes have a finitely generated free group as fundamental group and the groups X and Y are free factors of the appropriate fundamental group. Moreover the loops corresponding to α and β are now of length pq. So the first step in the proof can be applied here. The net result again is that G is a free factor of an automatic group and is consequently automatic in this case as well.

We are now almost in a position to begin the proof of Theorem D. We note first that by the work of Gromov [Gr] an infinite cyclic subgroup Z of a negatively curved group X is quasigeodesic. Hence by Theorem 1 of III.3, Z is rational. There are a number of ways to prove this fact. For instance it can be deduced as a consequence of the classification of isometries of a Cayley graph of a negatively curved group G [GH, 8.20] . Each isometry ϕ has a limit set $L(\phi)$ in the hyperbolic boundary δG which consists of at most two points. In particular, left translation by an element $z \in G$ is an isometry, which we denote by the same letter z. It is the case that $L(z)$ is empty if and only if z has finite order [GH,8.28], $L(z)$ consists of two points if and only if the subgroup $\langle z \rangle$ is quasigeodesic [GH, 8.21], and the "parabolic" case of precisely one point cannot occur [GH, 8.29].

An alternative proof can be obtained by using (1) implies (2) from Theorem 1 of III.3 and [GS3]. It follows from [GS3 4.4] and the fact that hyperbolic groups are biautomatic that an infinite cyclic subgroup $\langle z\rangle$ is a a subgroup of finite index in a direct factor of $C(C_G(z))$, the centre of the centralizer of z in G. Abelian subgroups of a hyperbolic group are finite extensions of a cyclic group ([Gr], [GS3,5.1]). But this means that $\langle z\rangle$ is a quasiconvex subgroup with respect to the language of geodesics.

It follows immediately then from Theorem C that if Y is a second negatively curved group containing Z, then the amalgamated product $G = X \star_Z Y$ is asynchronously automatic. Our objective is to prove that G is automatic. It suffices to prove that condition (3) holds, since this is the additional assumption of Theorem C that ensures an automatic structure for G.

Now Gromov's theorem that a cyclic subgroup of a negatively curved group is quasigeodesic implies that given a generator z and a finite generating set $\mathcal{X}$ for X, there are constants λ and ϵ such that for all n, $|z^n| \geq \lambda n - \epsilon$. We will show slightly more. We will show that there are constants $\tau = \tau_{\mathcal{X}}(\overline{z})$ and ϵ so that for all n, $n\tau - \epsilon \leq |\overline{z^n}|_{\mathcal{X}} \leq n\tau + \epsilon$. This τ is none other than the *translation length* of $\overline{z}$ (see below).

To this end, let us suppose henceforth that X, Y are negatively curved groups and that Z is an infinite cyclic subgroup of both X and Y. Let $\mathcal{X}$ be a finite set of group generators of X which we assume contains an element $\{z\}$ such that $\overline{z}$ generates Z. Put $\mathcal{D} = \mathcal{X} \cup \mathcal{X}^{-1}$. Then $(\mathcal{D}, L(X))$ is an automatic structure for X, where $L(X)$ is the set of all geodesic words over $\mathcal{D}$. We have already noted that if Z is an infinite cyclic subgroup of X, then Z is rational. So there is a regular set $L(Z) \subseteq L(X)$ which contains a representative for each element of Z. Now, following Gromov [Gr] (see also [GS3]) we define the *translation length* of an element $g \in X$ with respect to the generating set $\mathcal{X}$ to be

$$\tau_{\mathcal{X}}(g) = \lim_{n\to\infty} \frac{|g^n|_{\mathcal{X}}}{n}.$$

This limit exists as the sequence $|g^n|_{\mathcal{X}}$ is subadditive ([Ma]).

As Z is a rational subgroup of X, it follows from the Pumping Lemma, II.A.4, that there is a word $w = uyv$ in $L(Z)$ such that $uy^iv \in L(Z)$ for all $i \geq 0$.

Since the geodesics uy^iv are all of different lengths, they must represent distinct elements of Z. In particular $\overline{uyv} \neq \overline{uy^2v}$. Consequently

$$\overline{v^{-1}yv} = \overline{z^m}$$

for some non-zero integer m. We consider first the case where $m > 0$. The other case, $\overline{v^{-1}yv} = \overline{z^{-1}}^m$ $(m > 0)$, can be handled similarly.

Notice now that

$$\overline{v^{-1}y^jv} = \overline{z^{mj}}$$

for every positive integer j. Moreover y^{mj} is a geodesic since it is a subword of a geodesic. It follows that $|\overline{v^{-1}y^jv}|_{\mathcal{X}} \geq j\ell(y) - 2\ell(v)$. Therefore

$$j\ell(y) - 2\ell(v) \leq |\overline{z^{mj}}| \leq j\ell(y) + 2\ell(v).$$

Now if n is any positive integer, then $n = mj + r$ where $0 \leq r < m$. We then have

$$mj\ell(y) - 2\ell(y) - |\overline{z^r}| \leq |\overline{z^n}| \leq mj\ell(y) + 2\ell(y) + |\overline{z^r}|.$$

Since r is bounded, so is $|\overline{z^r}|$. In particular,

$$\tau_{\mathcal{X}}(\overline{z}) = \lim_{n\to\infty} \frac{|\overline{z^n}|_{\mathcal{X}}}{n} = \lim_{m\to\infty} \frac{|\overline{z^{mj}}|_{\mathcal{X}}}{mj} = \frac{\ell(y)}{m}.$$

Notice that $\tau_{\mathcal{X}}(\overline{z})$ is always rational.

Lemma 4. *Let X be negatively curved, and let Z be an infinite cyclic subgroup of X, generated by $z \in X$. Let $\mathcal{X}$ be a finite set of group generators of X. For any positive integer d, there is a set of group generators $\mathcal{X}'$ of X such that*

$$\tau_{\mathcal{X}}(z) = d\tau_{\mathcal{X}'}(z).$$

Proof. Let $\mathcal{X}'$ be the union of $\mathcal{X}$ together with a set in a one-to-one correspondence with the elements of X of length at most d. Then $\mathcal{X}'$ is a second set of group generators of X, where the generation map is defined in the obvious way. Notice that if $f \in X$ is of $\mathcal{X}$-length at least d, then

$$d|f|_{\mathcal{X}'} \leq |f|_{\mathcal{X}} \leq d|f|_{\mathcal{X}'} + 1.$$

So if $m > 0$ is given and $j > 0$ is chosen sufficiently large, then

$$d|\overline{z}^{mj}|_{\mathcal{X}'} \leq |\overline{z}^{mj}|_{\mathcal{X}} \leq d|\overline{z}^{mj}|_{\mathcal{X}'} + 1.$$

Dividing by mj and letting j go to infinity, we find that

$$d\tau_{\mathcal{X}'}(\overline{z}) = \tau_{\mathcal{X}}(\overline{z})$$

as required.

Lemma 5. *Let X, Y be negatively curved curved groups, and let Z, Z_1 be infinite cyclic subgroups of X and Y respectively. Let $\mathcal{X}$ be a finite set of group generators of X containing a generator z of Z and let $\mathcal{Y}$ be a finite set of group generators of Y containing a generator z_1 of Z_1. Then we can choose finite sets of group generators $\mathcal{X}'$ and $\mathcal{Y}'$ of X and Y respectively such that*

$$\tau_{\mathcal{X}'}(\overline{z}) = \tau_{\mathcal{Y}'}(\overline{z_1}).$$

Proof. Now $\tau_{\mathcal{X}}(\overline{z}) = r/s$ and $\tau_{\mathcal{Y}}(\overline{z_1}) = r_1/s_1$ are positive rational numbers. It follows from Lemma 3 that by choosing a new set $\mathcal{X}'$ of generators of X and a new set $\mathcal{Y}'$ of generators of Y we can arrange that

$$\tau_{\mathcal{X}'}(\overline{z}) = \frac{\tau_{\mathcal{X}}(\overline{z})}{rs_1}$$

and

$$\tau_{\mathcal{Y}'}(\overline{z_1}) = \frac{\tau_{\mathcal{Y}}(\overline{z_1})}{r_1 s}.$$

This completes the proof of Lemma 5.

Next we prove

Lemma 6. *Let G be the amalgamated product of two negatively curved groups X and Y amalgamating a cyclic subgroup Z. Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are finite sets of group generators of X and Y respectively and that both $\mathcal{X}$ and $\mathcal{Y}$ contain a letter z such that $\overline{z}$ generates Z. Suppose also that $\tau_{\mathcal{X}}(\overline{z}) = \tau_{\mathcal{Y}}(\overline{z})$. Let $L(X)$ be the set of all geodesic words over $\mathcal{X} \cup \mathcal{X}^{-1}$ and let $L(Y)$ be the set of all geodesic words* *over $\mathcal{Y} \cup \mathcal{Y}^{-1}$. Then there exists a positive constant k such that if $v \in L(X)$ and $w \in L(Y)$ are such that*

$$\overline{v} = \overline{w} = \overline{z}^n$$

for some n, then v and w are k-fellow travellers in the Cayley graph $\Gamma(G)$.

Proof. Suppose for the moment that $n > 0$ is a large integer and that v and w are the geodesic words given above. Now by the work of Gromov [Gr] we can find a constant K such that every geodesic in the Cayley graph $\Gamma_{\mathcal{X}}(X)$ of X which ends up at some power $\overline{z}^n$ of $\overline{z}$ stays within distance K of the path z^n and similarly for geodesics in the Cayley graph $\Gamma_{\mathcal{Y}}(Y)$ of Y. Notice that K is independent of n. So there exist functions $f(t)$ and $g(t)$ from $[0, \infty)$ to $\{0, \dots, n\}$ such that

$$d_{\mathcal{X}}(v(t), \overline{z}^{f(t)}) \leq K$$

and

$$d_{\mathcal{Y}}(w(t), \overline{z}^{g(t)}) \leq K$$

and

$$||\overline{z}|_{\mathcal{X}} - n\tau_{\mathcal{X}}(\overline{z})| \leq K$$

and

$$||\overline{z}|_{\mathcal{Y}} - n\tau_{\mathcal{Y}}(\overline{z})| \leq K$$

hold for all $n > 0$. Now since $\tau_{\mathcal{X}}(\overline{z}) = \tau_{\mathcal{Y}}(\overline{z}) = \tau$, we have for all $n > 0$

$$||\overline{z}^n|_{\mathcal{X}} - |\overline{z}^n|_{\mathcal{Y}}| \leq 2K$$

and so

$$vert f(t)\tau - g(t)\tau| \leq 4K.$$

It now follows that (using Γ to denote $\Gamma_{\mathcal{X}\cup\mathcal{Y}}(G)$)

$$\begin{aligned} d_\Gamma(v(t), w(t)) &\leq 2K + d_\Gamma(\overline{z}^{f(t)}, \overline{z}^{g(t)}) \\ &\leq 2K + M \end{aligned}$$

where

$$M = \max\{|\overline{z}^p|_{\mathcal{X}\cup\mathcal{Y}} \mid 0 \leq p \leq 4K/\tau\}.$$

Thus if we set $k = 2K + M$, the conclusion of Lemma 6 holds for positive n. The case $n < 0$ can be handled analogously, and so the proof of Lemma 6 is complete.

It follows, on appealing to Lemma 5 and Lemma 6 as needed, that condition (3) of Theorem A holds. This completes the proof of Theorem D.

These arguments can be made more general. In fact, with a little bit of work they can be made to show the following: If X and Y are negatively curved groups each containing the rational subgroup Z, and there are generating sets $\mathcal{X}$ and $\mathcal{Y}$ and a constant ϵ such that for any element g of Z, $||g|_{\mathcal{X}} - |g|_{\mathcal{Y}}| \leq \epsilon$, then condition (3) of Theorem A holds. In particular $G = X \star_Z Y$ is automatic. We leave the task of checking this to the reader.

6. Free factors and retracts.

Suppose that G is a group and that $\mathcal{X}$ is a finite set of group generators for G. Let $\mathcal{A} = \mathcal{X} \cup \mathcal{X}^{-1}$ and adopt the usual notation with $\mu : \mathcal{A}^\star \longrightarrow G$ mapping w to $\overline{w}$. If ϕ is a homomorphism of G onto a group H, then $\phi \circ \mu$ restricted to $\mathcal{X}$ is a group generation map for H. The homomorphism ϕ induces a graph homomorphism, also denoted by ϕ, of $\Gamma_{\mathcal{X}}(G)$ onto $\Gamma_{\mathcal{X}}(H)$. This implies that

$$d_G(g_1, g_2) \geq d_H(\phi(g_1), \phi(g_2)),$$

where d_G and d_H denote distances in the respective Cayley graphs. We shall make use of this remark shortly.

Our first objective is to prove the following theorem.

Theorem F. *If H is a free factor of an automatic (asynchronously automatic) group, then H is automatic (asynchronously automatic).*

Notice that such a free factor H is a retract of G, i.e. there is a homomorphism ϕ of G onto H which is the indentity on H.

In order to prove the theorem we begin by proving the following lemma.

Lemma 1. *Suppose that H is a retract of G and that $(\mathcal{A}, L)$ is an automatic structure (asynchronously automatic structure) for G. If H is L-rational, i.e. if*

$$L(H) = \{w \mid w \in L \textit{ and } \overline{w} \in H\}$$

is regular, then there is an automatic structure (asynchronously automatic structure) $(\mathcal{A}, L'(H))$ for H, with $L'(H) \subset L(H)$.

Proof. Notice first that by Proposition 1 of section II.B.1 (see section II.B.7 for the asynchronous case) there is an automatic structure $(\mathcal{A}, L')$ for G with $L' \subset L$, and which contains only a finite number of representatives for each element of G. Now H is also L'-rational in G as $L(H) \cap L' = L'(H)$ is a regular language.

We consider first the synchronous case. Let $\phi : G \longrightarrow H$ be a retraction from G to H. It suffices to prove that that $L'(H)$ has the k-fellow traveller property for some k. To this end, suppose that $u, v \in L'(H)$ and that $d_H(\overline{u}, \overline{v}) \leq 1$. Put $\mathcal{A} = \{a_1, \ldots, a_q\}$ and $a_0 = e$. Then

$$\phi(\overline{u}) = \phi(\overline{v})\phi(\overline{a_i}).$$

Hence

$$\overline{u} = \overline{v}\phi(\overline{a_i}).$$

Now $\phi(\overline{a_i})$ can be expressed as a word in the $\overline{a_i}$:

$$\phi(\overline{a_i}) = w_i(\overline{a_1}, \ldots, \overline{a_q}).$$

Hence

$$\overline{u} = \overline{v} w_i(\overline{a_1}, \ldots, \overline{a_q}).$$

So if ℓ is the maximum of the lengths of all of these $w_i(\overline{a_1}, \ldots, \overline{a_q})$, $i = 0, \ldots, q$, then there exists a k such that u, v are k-fellow travellers in $\Gamma_{\mathcal{X}}(G)$ and hence also in $\Gamma_{\mathcal{X}}(H)$.

The proof that $(\mathcal{A}, L'(H))$ is an asynchronously automatic structure for H if $(\mathcal{A}, L')$ is an asynchronously automatic structure for G with a finite number of representatives for each element can be carried out in much the same way, using instead the sharper version of Theorem 1 of II.B.7 described in the preamble to the proof of that theorem. Here we need only observe that if u and v and v and w are two pairs of asynchronous k-fellow travellers, then u and w are asynchronous $2k$-fellow travellers. Again we have to re-express each $\phi(\overline{a_i})$ as $w_i(\overline{a_1}, \ldots, \overline{a_q})$ and notice that $L'(H)$ has the $k\ell$-fellow traveller property.

We are now in a position to complete the proof of Theorem F. To this end let $G = H \star F$, and

$$\mathcal{A} = \{h_1, \ldots, h_m, f_1, \ldots, f_n\},$$

where $h_1, \ldots, h_m$ and $f_1, \ldots, f_n$ are finite sets of monoid generators for H and F respectively. Let $(\mathcal{A}, L)$ be an automatic (synchronously automatic) structure for G and suppose that every element of G has only finitely many representatives in L, and that the empty word e is a representative of the identity element. We now apply Lemma 3 of II.B.7 with $h = 0$. Let $z_1, \ldots, z_p$ be the set of "trivial infixes "guaranteed to be finite by Lemma 3 of II.B.7, i.e. the finitely many words z such that $uzw \in L$ for some choice of u, w and $\overline{z} = 1$. We now put

$$L(H) = L \cap \{h_1, \ldots, h_m, z_1, \ldots, z_p\}^\star.$$

Then $L(H)$ is regular. We claim that

$$L(H) = \{w \in L \mid \overline{w} \in H\}. \tag{1}$$

Let R denote the right-hand-side of (1). Then clearly $L(H) \subset R$. We have to prove the reverse inequality. Suppose that $w \in R$. Then w can be factored into one of the two forms

$$w = u_1 v_1 \ldots u_t v_t, \;\; w = v_1 u_1 \ldots v_t u_t$$

where $u_i \in \{h_1, \ldots, h_m, z_1, \ldots, z_p\}^\star$ and $v_i \in \{f_1, \ldots, f_n, z_1, \ldots, z_p\}^\star$ and t is chosen to be minimal. Notice that we allow $t = 0$ in which case $w = e$. We now show that $t = 1$. As $\overline{w} \in H$, there is a word $w_1 \in \{h_1, \ldots, h_n\}^\star$ such that $\overline{w_1} = \overline{w}^{-1}$. If w has the first of the two forms possible, let w' be the word obtained by replacing u_1 by $w_1 u_1 = u_1'$. Then $\overline{w'} = 1$; if $\overline{u_1'} = 1$, then consider the word $w' = v_1 u_2 \ldots v_t$.

Thus by the definition of free products, some $\overline{u_i} = 1$ or some $\overline{v_i} = 1$. But this contradicts the minimality of expression of the word w in terms of u_i and v_i. Thus $w = u_1$.

The second case, i.e. when $w = v_1 u_1 \ldots v_t u_t$ follows similarly, this time post-multiplying by w_1.

This completes the proof of Theorem F.

Next we prove the

Theorem G. *Let G be a finitely presented group and let the subgroup H of G be a retract of G. If G satisfies a linear or a quadratic or an exponential isoperimetric inequality, then H satisfies correspondingly a linear or a quadratic or an exponential isoperimetric inequality.*

Proof. We shall establish during the course of the argument that H is also finitely presented (a result due to Wall).

Let $\{x_1, \ldots, x_m\}$ be finite set of generators for G, and let $r : G \to H$ be a retraction. If m is the number of distinct images $r(x_i)$, we can consider $\{y_1, \ldots, y_m\}$ as a set of generators for H, using the natural map $\mu(y_j) = r(x_i)$ for some i (as r is onto). Now consider a finite presentation

$$\mathcal{P} = \langle x_1, \ldots, x_m, y_1, \ldots, y_n \mid r_1, \ldots, r_k \rangle$$

for G. For $w \in F(y_1, \ldots, y_n)$ and $\overline{w} = 1$ in H, then $\overline{w} = 1$ in G too. Hence there are words $p_i \in F(x_1, \ldots, x_m, y_1, \ldots, y_n) = F$ such that $w = \prod_{i=1}^{N} p_i s_i^{\pm} p_i^{-1}$ in F, where $s_i \in \{r_1, \ldots, r_k\}$. The map r restricted to the set of generators induces a map $\rho : F \to F(y_1, \ldots, y_n)$. But then

$$\rho(w) = w = \prod_{i=1}^{N} \rho(p_i)\rho(s_i)^{\pm}\rho(p_i).$$

It follows that

$$\mathcal{P}' = \langle y_1, \ldots, y_n \mid \rho(r_1), \ldots, \rho(r_k) \rangle$$

is a finite presentation for the group H, and it is clear that the Dehn function for the presentation $\mathcal{P}$ of G is a Dehn function for the presentation $\mathcal{P}'$ of H.

In particular, then, a retract of a negatively curved group is also negatively curved and therefore, in particular, a free factor of a negatively curved group is also negatively curved.

In fact it is possible to generalise Theorem G a little in the case of negatively curved groups and automatic groups.

Theorem H. *Suppose that $G = X \star_Z Y$ where Z is finite. Then G is negatively curved (automatic) if and only if both X and Y are negatively curved (automatic).*

The proof of Theorem H follows much the same lines as that of Theorem G. We start out by observing that because Z is finite, G is finitely presented if and only if both X and Y are finitely presented (Karrass and Solitar [KS]).

Now suppose that X and Y satisfy a linear isoperimetric inequality. We sketch the proof that G also satisfies a linear isoperimetric inequality. Suppose that $Z = \{z_1, \ldots, z_p\}$ and that $\mathcal{U} = \{u_1, \ldots, u_p\}$, $\mathcal{V} = \{v_1, \ldots, v_p\}$ are sets in one-one correspondence with Z. We choose finite presentations

$$X = < \mathcal{X}; R >, \ Y = < \mathcal{Y}; S >$$

for X and Y respectively so that $\mathcal{X} \supseteq \mathcal{U}$, $\mathcal{Y} \supseteq \mathcal{V}$ and the respective group generating maps send u_i to z_i and v_i to z_i. Let F be the free group on $\mathcal{W} = \mathcal{X} \cup \mathcal{Y}$. The group generating maps for $\mathcal{X}$ and $\mathcal{Y}$ extend to a group generating map of $\mathcal{W}$ into G. This gives rise to a homomorphism ϕ of F onto G with kernel K, say. Since

$$G = < \mathcal{W}; R \cup S \cup \{u_i v_i^{-1} \mid i = 1, \ldots, p\} >,$$

K is the normal closure in F of

$$\mathcal{Q} = R \cup S \cup \{u_i v_i^{-1} \mid i = 1, \ldots, p\}.$$

We have to prove that if $w \in K$ is of length n, then w can be expressed as a product of cn conjugates of the elements of $\mathcal{Q}$ and their inverses, where c is a constant independent of w. Now w breaks up into a product of subwords over $\mathcal{X}$ and $\mathcal{Y}$. At least one of these subwords, say the $\mathcal{X}$-word w', must have a value in Z, i.e. $\overline{w'} = \overline{u_i}$ for some i. So the relator $w'u_i^{-1}$ can be expressed as a product of $c_1 j$ conjugates of elements of R and their inverses, where j is the length of w' and c_1 is the constant guaranteed by the fact that the given presentation for X satisfies a linear isoperimetric inequality. So the subword w' of w can be replaced by v_i as a consequence of $c_1 j + 1$ relations. The resulting word w_1, say, is now expressed as a product of fewer $\mathcal{X}$ and $\mathcal{Y}$ subwords and so the process can be continued by an induction, thereby completing the proof. In fact we have proved somewhat more, namely that a free product of two groups amalgamating a finite subgroup satisfies the maximum of the isoperimetric inequalities satisfied by the factors.

Now suppose that the group G is negatively curved. As we remarked at the outset both X and Y are then finitely presented since G is finitely presented. We choose a group generating set $\mathcal{W}$ as detailed above. Our objective now is to sketch the proof that every geodesic triangle T in the Cayley graph $\Gamma_{\mathcal{X}}(X)$ of X is $\delta-thin$ for some δ. The three vertices of T define a geodesic triangle T' in the Cayley graph $\Gamma_{\mathcal{W}}(G)$ of G, which is therefore $\delta-$thin for some δ. The word w labelling a side of T' represents an element of X. Since $\mathcal{X}$ contains a "representative" u_i for each element of Z one can show that w can be replaced by a second geodesic word in $\Gamma_{\mathcal{W}}(G)$ with the same extremities as w with letters coming only from $\mathcal{X}$. This gives rise to a second geodesic triangle T'' in $\Gamma_{\mathcal{W}}(G)$ with the same vertices as T', whose sides are labelled by words over $\mathcal{X}$. This implies that T is actually a geodesic triangle in $\Gamma_{\mathcal{W}}(G)$ and consequently $\delta-$thin in $\Gamma_{\mathcal{W}}(G)$ and therefore, by a similar argument, also $\delta-$thin in $\Gamma_{\mathcal{X}}(X)$. This completes the proof.

Suppose now that G is the amalgamated product in the statement of the theorem and that $(\mathcal{X}, L(X))$ and $(\mathcal{Y}, L(Y))$ are automatic structures with uniqueness for the automatic groups X and Y respectively. We assume that the sets $\mathcal{U}$ and $\mathcal{V}$ (above) are contained in $\mathcal{X}$ and $\mathcal{Y}$ respectively, and that they are respectively the representatives in $L(X)$ and $L(Y)$ mapping onto Z. Let $\mathcal{W} = \mathcal{X} \cup \mathcal{Y}$. Our objective is to concoct an automatic structure over $\mathcal{W}$ for G. Let

$$\begin{aligned} L' = &\{w \mid w \in L(X) \text{ and for all } u \in \mathcal{U} \\ &\text{if } v \in L(X) \text{and } \overline{v} = \overline{wu}, \text{ then } w \leq v\}, \end{aligned}$$

where here $\leq$ is the lexicographic ordering introduced in II.A.5. It follows from the same argument used to prove Proposition 1 of II.B.1 that L' is regular. It follows then that $L(X/Z) = L' - \mathcal{U}$ is also regular. The corresponding set $L(Y/Z)$ contained in $L(Y)$ is then also regular. Consequently

$$L' = (L(Y/Z) \cup \{e\})(L(X/Z)L(Y/Z))^{\star}(L(X/Z) \cup \{e\})$$

is also regular and therefore so too is

$$L = L'\mathcal{U}.$$

It follows that $(\mathcal{W}, L)$ is a rational structure with uniqueness for G. In fact we claim that $(\mathcal{W}, L)$ is an automatic structure for G.

Now $L_= = \Delta(L)$ and so it is regular. We are left with checking that the sets L_w are regular for each $w \in \mathcal{W}$. Suppose first that $w = x \in \mathcal{X}$. Then for each u_i there is a word $w_{i,x} \in L(X)$ such that

$$\overline{u_i x} = \overline{w_{i,x} u_{i(x)}},$$

where here $i(x) \in \{1, \dots, p\}$. Now if v is any word over $\mathcal{X}$, then

$$L(X)_v = \{\nu(w_1, w_2) \mid w_1, w_2 \in L(X), \overline{w_1} = \overline{w_2 v}\},$$

is regular, by Lemma 1 of II.B.1. It follows that L_x is the union of the finitely many regular sets

$$\Delta(L')(L(X)_x) \cap ((L(X/Z) \times L(X/Z))(u_{i(x)}, u_i)$$

with i ranging over the integers from 1 to p. So L_x is regular. A similar argument applies to the other case where $w \in \mathcal{Y}$. So this completes the proof that G is automatic.

Finally, let us assume that the group G is automatic. Let $\mathcal{W}$ be the set of monoid generators for G chosen above. Now there is an automatic structure $(\mathcal{W}, L)$ with uniqueness for G. Now by Lemma 3 of II.B. 7 the uniqueness of representation for the elements of G implies that there are at most a finite number of subwords z with $xzy \in L$ and $\overline{z} \in Z$. Let $\mathcal{S} = \{z_1, \dots, z_r\}$ be the set of all such subwords z. Consequently

$$L(X) = \{w \mid w \in L \text{ and } \overline{w} \in X\} = L \cap (\mathcal{X} \cup \mathcal{S})^\star$$

is regular and contains a unique representative for each element of X. We supplement $\mathcal{X}$ by the disjoint set $\{a_0, \dots, a_r\}$ and extend the generation map to these new generators by putting $\{\overline{a_0} = 1, \overline{a_1} = \overline{z_1}, \dots, \overline{a_r} = \overline{z_r}\}$. We now replace each occurrence of each z_i in every word in $L(X)$ by $z_i z_0 \dots z_0$, where the number of z_0-s tacked on is arranged so that all of these words have exactly the same length. Let L'' be the resulting language. Then by a now familiar argument (see, e.g. the proof of Theorem 1 II.B. 3), the language L'' is regular. Moreover k is the fellow traveller constant of the language L of G and if k' is the length of the longest element in $\mathcal{S}$, then kk' is a fellow traveller constant for the language L''. So X is automatic, as claimed.

If we assume only that G is asynchronously automatic, then again it follows in similar vein that the factors X and Y are asynchronously automatic. We have not been able to carry out the converse in the case of aysnchronously automatic groups. The stumbling block for us has been our inability to concoct the counterpart to $L(X/Z)$ in this case.

Finally we can prove, using Britton's Lemma (see e.g. [LS]) in place of the normal form that we have been using in the case of amalgamated products, the following

Theorem I. *Let G be an HNN extension with a single stable letter with base B. Suppose that the associated subgroups are finite. Then G is negatively curved (automatic) if and only if B is negatively curved (automatic).*

The proof is similar to the one above and we will not give the details here.

7. Some examples.

Example 1

Let X be a negatively curved group, Z a quasiconvex subgroup of X for the language of all geodesics in the Cayley graph of X. Then the double $X \star_Z X$ is automatic. This is immediate from Theorem C, as words in $L(Z)$ run at the same rate in both sides of the amalgam.

Example 2

Let X and Z be as above. Then the HNN extension $X\star_Z$ is automatic. Here the HNN extension is formed using the inclusion homomorphism for both maps $Z \to X$. In this case $X\star_Z = X \star_Z Y$ where $Y = Z \times \mathbb{Z}$. The result now follows from Theorem A. Here one uses the language $L_1(Z)$ for Z obtained by shadowing the language $L(Z) \subset L(X)$. The language for Y is the product language $L(\mathbb{Z})L_1(Z)$, where $L(\mathbb{Z})$ is the language of all geodesics in the Cayley graph of $\mathbb{Z}$. Since $L_1(Z)$ sits in both sides of the amalgam and since shadowing runs at the same rate, condition (3) holds. Condition (4) on the X side of the amalgam follows from III.4.Lemma 5, and on the Y side follows from $L(Y/Z) = L(\mathbb{Z})$ (a little care is required here since Y is not negatively curved).

Example 3

It is immediate from I.10.Corollary E1 that if F is a finitely generated free group and if $\phi : F \to F$ is an automorphism, then the split extension $G = F \otimes_\phi \mathbb{Z}$ is asynchronously automatic. It is an interesting open question whether G is synchronously automatic. We note in this connection that the authors of [BF] maintain that if ϕ has no non-trivial periodic conjugacy classes of elements of F, then G is negatively curved.

Example 4

Contrary to the case of cyclic amalgams, III.5 above, an example of an amalgam of two finitely generated free groups amalgamating a finite index subgroup of each need not be automatic (although by Theorem E it is always asynchronously automatic). An example can be found in [Ge, §6]; the proof goes by showing that the quadratic isoperimetric inequality fails to hold. In this context we mention a positive result. Suppose that $f_i : \Gamma \to \Gamma_i$ are immersions of finite connected graphs, $i = 1, 2$ and x is a vertex of Γ.[St] We can then form the amalgam

$$G = \pi_1(\Gamma_1, f_1) \star_{\pi_1(\Gamma, x)} \pi_1(\Gamma_2, f_2(x))$$

where the injections are induced by f_i. Then G is synchronously automatic. The argument is geometric and proceeds along the lines of the proof of Corollary 1 to Theorem D of III.5 above. One shows that G possesses a C(4)–T(4) presentation and hence is automatic [GS2].

We now construct some non-automatic groups.

The proofs that the groups that we will construct here do not satisfy certain isoperimetric inequalities depend on the use of *disc diagrams* that we have already alluded to before. With this in mind we begin this section with a discussion of such diagrams. It is useful here to distinguish a group from its presentations and we will do so in this section.

Given a presentation $\mathcal{P} = \langle X; R\rangle$ of a group G, there is a naturally associated 2-complex $K(\mathcal{P})$ with one vertex, one 1-cell for each element of X, and one 2-cell $D(r)$ for relator $r \in R$. The attaching map for the 2-cell $D(r)$ identifies the boundary of the cell with a closed curve in the 1-skeleton $K(\mathcal{P})^{(1)}$, based at the vertex, and representing the word r in the free group F on X. Given a word $w \in F$ such that $\overline{w} = 1$ in G, there is a map of a disc $(D, \partial D)$ to $(K(\mathcal{P}), K(\mathcal{P})^{(1)})$ such that ∂D represents the word w in $F(X)$. After a homotopy, the cell structure of $K(\mathcal{P})$ induces a cell decomposition of a simply connected complex D, consisting of discs joined by arcs. The 1-cells of D can be labelled and oriented by their images in K. Reading in order the labels on the edges on the boundary of each 2-cell of D spells out a cyclic conjugate of a word in $R \cup R^{-1}$, and reading from an appropriate vertex and in the right direction, the labels on the boundary of D spell out the word w. We call D a *singular disc diagram* for $w = 1$ in $\mathcal{P}$, and we call the 1-cells of D *edges*, and the 2-cells *faces*. (For more information on disc diagrams see [LS, chapter V], or [GS1].)

A disc diagram D for $w = 1$ in G is called *minimal* if any diagram D$'$ for $w = 1$ has at least as many regions as D.

Notice that if there is a singular disc diagram for $w = 1$ in G which has N faces, then w can be expressed as a product on N conjugates of elements of R. Thus a Dehn function (see II.B.5) can be regarded as a function giving an upper bound for the number of faces in a minimal singular disc diagram for $w = 1$ in terms of the length of the word w.

We prove first the

Proposition 1. *If $\mathcal{P}_1$ and $\mathcal{P}_2$ are finite presentations of the group G, and f_1 is a Dehn function for $\mathcal{P}_1$ then there exists a Dehn function f_2 for $\mathcal{P}_2$ and constants K, K', K'' such that for every non-negative integer n*

$$f_2(n) \leq K' f_1(Kn) + K''n.$$

In particular, if f_1 is a polynomial of degree $d > 0$, then then there is polynomial Dehn function of degree d for $\mathcal{P}_2$ and if f_1 is exponential, then f_2 can also be chosen exponential.

Proof. Let $\mathcal{P}_1 = \langle X_1; R_1 \rangle$ and $\mathcal{P}_2 = \langle X_2; R_2 \rangle$ be finite presentations for G and let F_1 and F_2 be respectively free groups on X_1 and X_2. Let w be a word in F_2 which represents the trivial element of G. For each generator x_i in X_2, there is a word $\phi(x_i)$ in F_1 representing the same element of G. We show how to obtain a diagram for $w = 1$ in $\mathcal{P}_2$. First translate the word w into a word $\phi(w)$ in the generators X_1, by using the map $x_i \mapsto \phi(x_i)$. If K is the maximum length of the words $\phi(x_i)$, then the length of $\phi(w)$ is at most $K\ell(w)$. It follows that there is a diagram D_1 for $\phi(w) = 1$ in $\mathcal{P}_1$ with at most $f_1(K\ell(w))$ faces. On the other hand there is a map ψ, the counterpart to ϕ, which expresses each generator y_j of X_1 as a word $\psi(y_j)$ in the generators X_2. Relabel each edge of the diagram D_1 by applying the map ψ to each label. The labels on the faces are translated into possibly unreduced words in F_2.

Each of these words represents the trivial element of G, and there are only a finite number of words which can occur, corresponding to the finitely many relators in R_1.

So there is a singular disc diagram for each face of D_1, and if the greatest number of faces occurring in one of these is K' then we have found a diagram D_2 with at most $K'f(K\ell(w))$ faces for $\psi(\phi(w)) = 1$ in $\mathcal{P}_2$. To make a diagram bounded by the word w, we require some new faces corresponding to $x_i = \psi(\phi(x_i))$ for each generator $x_i \in X_2$. Each of the words $\psi(\phi(x_i))x_i^{-1}$ can be written as a product of conjugates of relators in R_2. Let K'' be the maximum number of conjugates required, i.e. K'' is the maximum number of faces required for a diagram for $\psi(\phi(x_i))x_i^{-1} = 1$ in $\mathcal{P}_2$.

It follows therefore from Proposition 1 that if one finite presentation of a group G satisfies a *linear, quadratic, cubic, etc. isoperimetric inequality* then every other finite presentation satisfies a like isoperimetric inequality. So the existence of such isoperimetric inequalities for a finitely presented group is independent of the choice of finite presentations, i.e. is a property of the group itself.

The change of presentation induces quasi-isometry between Cayley graphs. For more recent results on quasi-isometries and isoperimetric inequalities in a broader context, see [**A**].

Finally, suppose that the group G is given by the finite presentation $G =< X; R >$. Then if w is a relator in G, it follows from the construction of a disc diagram for w that if f is a Dehn function for this presentation of G and if F is the free group on X, then

$$w = \prod p_i r_i^{\pm 1} p_i^{-1},$$

where $r_i \in R, p_i \in F$ and $\ell(p_i) < f(\ell(w))K + \ell(w)$ (cf. [LS]).

Our first objective here is to prove that the finitely presented group G given by the finite presentation

$$\mathcal{Q} = \langle a, b, t, u; tat^{-1} = ab, tbt^{-1} = a, uau^{-1} = ab, ubu^{-1} = a \rangle$$

is not automatic even though it is asynchronously automatic. In fact we shall show that this presentation does not even satisfy a quadratic isoperimetric inequality. Gersten [G] has obtained a number of further examples of asynchronously automatic groups which are not automatic. In particular he has proved that the groups

$$G_{n,m} = \langle x, y; yx^n y^{-1} = x^m \rangle$$

introduced by Baumslag and Solitar [BS] fail to be automatic when $|n| \neq |m|$. In fact these goups do not even satisfy a quadratic isoperimetric inequality.

We use the ideas described above to show that the presentation $\mathcal{Q}$ does not satisfy a quadratic isoperimetric inequality. First we prove the following lemma.

Lemma 1. *$K = K(\mathcal{Q})$ is aspherical.*

Proof. The group G is built up from the free group F on a, b by adjoining the generators t, u which both act on F by the automorphism $a \to ab$, $b \to a$. The resulting complex K is thus the union of two mapping tori over the wedge of two circles ("figure of eight curve") $\mathcal{R}$, identifying the two copies of $\mathcal{R}$. The inclusion of $\mathcal{R}$ in the space is an injection on π_1 so by a theorem of J.H.C.Whitehead, K is aspherical.

Definition A connected 2-complex K is called *Cockcroft* if the Hurewicz homomorphism $\pi_2(K) \to H_2(K)$ is trivial.

Clearly an aspherical 2-complex is Cockcroft.

Definition A disc diagram D is called *positive* if, for each 2-cell of K, the orientations on each region of D mapping to this 2-cell induce the same orientation on D.

Proposition 2. *If K is a Cockcroft 2-complex then any positive disc diagram is minimal.*

Proof. Let $h : D \to K$ be a positive disc diagram, and let $h' : D' \to K$ be some other disc diagram, for $w = 1$ in G. Form a map $g : h \sqcup -h' : S^2 \to K$ by gluing together the two discs D, D' along their boundaries. Then g represents a class in $\pi_2(K)$ which is 0 in $H_2(K)$. But as K is a 2-complex, $H_2(K) = Z_2(K)$, the group of 2-cycles. Letting $c(h), c(h')$ denote the chains determined by h and h' in the free abelian group $C_2(K, \mathbb{Z})$, it follows that $c(h) = c(h')$. As h is positive, the coefficient of the 2-cell σ in $C_2(K, \mathbb{Z})$ is the number of regions in D mapping to σ in K. This means that the number of regions in D' is at least the number of regions in D.

We now apply this criterion to the 2-complex $K(\mathcal{Q})$ above, and a particular class of words. Let $w_n = [u^{-n}t^n, a]$ (where $[x, y]$ denotes the commutator $xyx^{-1}y^{-1}$). This is a freely reduced word of length 4n+2, and $w_n = 1$ in G, as can be seen from the positive disc diagram exhibited below (the case n=3).

Figure 1: A positive diagram for $w_3 = 1$

The number of regions f_n in such a diagram increases exponentially with n. The diagram for w_{n+1} is obtained from the diagram for w_n by adding $2(a(n)+b(n))$ new regions, where $a(n), b(n)$ denote the number of edges labelled a, b in the X-axis of the diagram. Note that $a(n+1) = a(n) + b(n)$ and $b(n+1) = a(n)$, so f_n grows like the Fibonacci numbers, which grow like $(\frac{1+\sqrt{5}}{2})^n$.

This shows that G, which is an HNN extension of a free group on two generators with two stable letters (see [LS]) and hence aysnchronously automatic, does not satisfy a quadratic isopermetric inequality and is therefore not an automatic group (cf. the remarks in I.10).

We give next an example of an amalgamated product $G = A \star_Z Y$ of two automatic groups X and Y with a finitely generated subgroup Z amalgamated which is not automatic. The amalgamated subgroup $Z =< a, b; aba^{-1}b^{-1} >$ is free abelian of rank two.

In order to define X and Y, let ϕ and ψ be the automorphisms of Z represented respectively by the matrices

$$\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \text{and} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

These two matrices are of orders 6 and 4 respectively and generate $SL_2(\mathbb{Z})$. This means that every element of $SL_2(\mathbb{Z})$ can be expressed as a product of positive powers of ϕ and ψ. In particular

$$M = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} = p(\phi, \psi)$$

is such a product $p(\phi, \psi)$ of positive powers of ϕ and ψ. Now let X and Y be the split extensions of Z defined respectively by the automorphisms ϕ and ψ:

$$X =< a, b, x; ab = ba, xax^{-1} = \phi(a) = ab, xbx^{-1} = \phi(b) = a^{-1} >$$

$$Y =< a, b, y; ab = ba, yay^{-1} = \psi(a) = b, yby^{-1} = \psi(b) = a^{-1} > .$$

The groups X and Y are fundamental groups of closed orientable three-manifolds, fibering over a circle, with torus fiber. As the two automorphisms are of finite order, both X and Y contain a free abelian group of rank three as a subgroup of finite index. Hence they are both automatic.

Our objective is to prove the

Proposition 3. *The amalgamated product $G = X \star_Z Y$ is not an automatic group.*

Let us put

$$d = p(x, y).$$

Thus d acts by conjugation on Z by the matrix M above. Now let us put

$$w_n = [d^n a d^{-n}, a] (n = 1, 2, \dots).$$

Then each w_n is a relator in G. Now each w_n, viewed as an element in the free group F on a, b, x, y, is of length $4n\delta + 4$, where δ is the length of the word $p(x, y)$ representing d. We shall prove that a minimal diagram for $w_n = 1$ has a number of faces which grows exponentially with n, and therefore G is not an automatic group.

As there is a bound on the number of edges in each face of a diagram for a finite presentation, namely 5 here, it suffices to show that in a diagram D_n for $w_n = 1$ there is an imbedded arc in the 1-skeleton consisting of a number of edges which is bounded below by a function which grows exponentially with n. We obtain such an arc as follows.

Now map G to $\mathbb{Z}$ by sending a and b to 0 and x and y to 1. Since $w_n = 1$ in G, the composite map $f_n : D_n \longrightarrow K(G) \longrightarrow S^1$ lifts to a map $\tilde{f}_n : D_n \longrightarrow \mathbb{R}$. In view of the fact that d is a product of positive powers of x and y, on the boundary of D_n the function $\tilde{f}_n$ achieves its maximum value N_n, say and its minimum value 0, on two edges labelled $\alpha^{\pm}$, along which the function is constant. Call these the maximimum and minimum edges. The function is monotonic between these four edges because d is a product of positive powers of x and y. Now notice that at non-integer values of t in the range of $\tilde{f}_n$, the level sets $\tilde{f}_n^{-1}(t)$ are embedded 1-manifolds because of the form of the relations and they meet the boundary in exactly four points. The arc components (the closed loops in a level set do not interest us here) thus fall into two types, according to whether they separate the maximum edges (Type 1) or not (Type 2).

Figure 2: The two types of embedded arcs in level sets

There are three cases to consider, viz.:

(1) all arcs are of type 1;
(2) all arcs are of type 2;
(3) there are arcs of both types.

We consider the three possibilitites in turn.
(1) Here there is an arc A in the 1-skeleton of D_n connecting thje two maximum edges, and the label on this path must be the word $d^{-n}ad^n$. Since d acts on Z essentially by M and the entries of the matrices $M^{\pm n}$ grow exponentially with n, the arc A has the required property.
(2) In this case, as in the previous one, there is an arc A in the 1-skeleton of D_n joining the minimum edges, with label d^nad^{-n} and this arc has the required property.
(3) In D_n a change occurs between type 1 and type 2 arcs at an integer level i, say. This can only happen for one value of i, because of the form of $\tilde{f}_n$ on δD_n.

Figure 3

If $i \geq N_n/2$, then there is a level arc in the 1-skeleton of D_n joining points on δD_n, whose label is

$$d^{-n}ad^n, \text{ where } m = [(n-1)/2]$$

and the number of edges in this arc is an exponential function of n. Similarly if $i < N_n/2$. This completes the proof of Proposition 2.

This last example suggest two questions which we have not yet been able to answer.

(1) If X and Y are automatic groups and Z is free abelian of rank two, is $X \star_Z Y$ asynchronously automatic?

(2) If X and Y are fundamental groups of 3-manifolds with incompressible torus boundaries and if Z is the peripheral subgroup of X and also of Y, is $X \star_Z Y$ automatic?

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